TEMPERLANGAN KAPUTO OPERATORI QATNASHGAN SUB-DIFFUZIYA TENGLAMASI UCHUN BOSHLANG‘ICH-CHEGARAVIY SHARTLI MASALA

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Annotatsiya: *Ushbu maqolada temperlangan Kaputo operatori qatnashgan sub-diffuziya tenglamasi uchun boshlang‘ich-chegaraviy shartli masalaning yechimi yaqqol topilgan.*

Kalit so'zlar: *Bessel tengsizligi, Koshi-Bunyakovskiy tengsizligi, temperlangan Kaputo kasr tartibli hosila operatori.*

УСЛОВНАЯ ЗАДАЧА С НАЧАЛЬНО ГРАНИЧНЫМИ УСЛОВИЯМИ ДЛЯ УРАВНЕНИЯ СУБДИФФУЗИИ С ТЕМПЕРИРОВАННЫМ ОПЕРАТОРОМ КАПУТО

Аннотация: *В данной статье показано явное решение задачи с начально граничными условиями для уравнения субдиффузии с темперированным оператором Капуто.*

Ключевые слова: *Неравенство Бесселя, неравенство Коши-Буняковского, дробный оператор производной Капуто.*

THE INITIAL-BOUNDARY CONDITIONAL ISSUE FOR THE SUB-DIFFUSION EQUATION IN WHICH THE TEMPERED CAPUTO OPERATOR IS INVOLVED

Abstract: *The solution to the initial-boundary conditional issue for the sub-diffusion equation in which the tempered Caputo operator is involved is evident in this article.*

Keywords: *Bessel’s inequality, Cauchy-Bunyakovsky’s inequality, tempered fractional Caputo derivative operator.*

$\Omega=\{(x,t):0<x<1,0<t<T\}$ sohada

$$ {}^{TC}D_{0t}^{\alpha,\gamma}U(x,t)=U_{xx}(x,t) \qquad (1)$$

tenglamani qaraymiz, bu yerda ${}^{TH}D_{0t}^{\alpha,\gamma}$ – temperlangan Kaputo kasr tartibli hosila operator

$$ {}^{TH}D_{0t}^{\alpha,\gamma}y(t)=\frac{e^{-\gamma t}}{\Gamma(1-\alpha)}\int_0^t(t-s)\frac{d}{ds}[e^{\gamma s}y(s)]ds$$

ko‘rinishda aniqlangan, $0<\alpha<1$, $\gamma\ge 0$.

A masala. Ω sohada (1) tenglamani va

$$U(x,0)=\varphi(x),\ x\in[0,1] \qquad (2)$$

$U(0,t)=0,\ U(1,t)=0,0<t<T$ (3)

boshlang‘ich, chegaraviy shartlarni qanoatlantiruvchi $U\in C(\bar{\Omega})$, $U_{xx}\in C(\Omega)$, ${}^{TC}D_{0t}^{\alpha,\gamma}\in C(\Omega)$ sinfga tegishli yechimi topilsin, bu yerda $\varphi(x)-$ berilgan funksiya.

Masalaning yechimini

$U(x,t)=X(x)T(t)$ (4)

ko‘rinishda qidiramiz.

(4) tenglikda $X(x)$ ni faqat x ga, $T(t)$ ni esa faqat t ga bog‘liq deb hisoblaymiz. (4) ning o‘ng tomonini (1) tenglamaning ko‘rinishiga qo‘yib,

${}^{TC}D_{0t}^{\alpha,\gamma}X(x)T(t)=X''(x)T(t)$ (5)

$\frac{{}^{TC}D_{0t}^{\alpha,\gamma}T(t)}{T(t)}=\frac{X''(x)}{X(x)}$ (6)

tenglamani hosil qilamiz. (6) tenglama uchun

$\frac{{}^{TC}D_{0t}^{\alpha,\gamma}T(t)}{T(t)}$ va $\frac{X''(x)}{X(x)}$ miqdorlarning har biri mos ravishda x ga ham t ga ham bog‘liq emas, ya’ni ular o‘zgarmas songa teng. Bu o‘zgarmasni $(-\lambda)$ orqali belgilab olamiz.

U holda (6) ga asosan

${}^{TC}D_{0t}^{\alpha,\gamma}T(t)+\lambda T(t)=0,t>0$ (7)

$X''(x)+\lambda X(x)=0,0<x<1$ (8)

tenglamani hosil qilamiz. (4) ko‘rinishdagi masalaning yechimini (3) shartlarni qanoatlantiruvchi, aynan nolga teng bo‘lmagan $U(x,t)$ yechimni topish uchun (8) tenglamaning

$X(0)=X(1)=0$ (9)

chegaraviy shartlarni qanoatlantiruvchi aynan nolga teng bo‘lmagan yechimni topish kerak.

Endi λ ning qabul qilishi mumkin bo‘lgan qiymatlarini uchta holatda qaraymiz.

1-hol. $\lambda<0$ bo‘lsin. U holda (8) tenglamaning umumiy yechimi

$X(x)=C_1e^{\sqrt{-l}x}+C_2e^{-\sqrt{-l}x}$

ko‘rinishga ega bo‘ladi. Bunda C_1 va C_2 - ixtiyoriy o‘zgarmaslar. (9) chegaraviy shartlarga asosan

$C_1+C_2=0,\ C_1e^{\sqrt{-l}}+C_2e^{-\sqrt{-l}}=0.$

bu sistemaning determinant noldan farqli bo‘lgani uchun $C_1=C_2=0.$

Demak, $X(x)\equiv 0.$

2-hol: $l=0$ bo‘lsin. Bu holda (8) tenglamaning umumiy yechimi quyidagicha bo‘ladi

$X(x)=C_1+C_2x$

(9) chegaraviy shartlarni qanoatlantirib $C_1=0,\ C_1+C_2=0$ tengliklarni hosil qilamiz. Bundan $C_1=0,\ C_2=0$, demak $X(x)\equiv 0$ ekanligi kelib chiqadi.

3-hol. $\lambda > 0$ bo‘lsin. Bu holda (8) tenglamaning umumiy yechimi

$$X(x) = C_1 \cos\sqrt{\lambda}\, x + C_2 \sin\sqrt{\lambda}\, x \qquad (10)$$

ko‘rinishga ega bo‘ladi. (9) chegaraviy shartlarga binoan

$$C_1 = 0, C_2 \sin\sqrt{\lambda} = 0. \qquad (11)$$

Biz $C_2 \neq 0$ deb hisoblaymiz, aks holda $X(x) = 0$ bo‘lib qoladi. Demak,

$$\sin\sqrt{\lambda} = 0$$

bo‘lgan holda, ya’ni $\sqrt{\lambda} = \pi n$ yoki $\lambda = \pi^2 n^2$ bo‘lganda, bu yerda n- natural son, $\{(8),(9)\}$ masala (11) ko‘rinishdagi aynan noldan farqli yechimga ega bo‘ladi.

Endi biz quyidagi xulosaga kelishimiz mumkin $\lambda_n = \pi^2 n^2, n = 1,2,3,...$

sonlar $\{(8),(9)\}$masalaning xos qiymatlaridir. $c_n \sin\pi nx$ funksiyalar esa (c_n - noldan farqli ixtiyoriy o‘zgarmas) ularga mos xos funksiyalardir. $c_n = 1$, $n = 1,2,3,...$ deb hisoblaymiz.

Masalaning yechimini

$$U_n(x,t) = \sum_{n=1}^{+\infty} T_n(t) X_n(x) \qquad (12)$$

ko‘rinishda qidiramiz.

(12) tenglikni (4) tenglama va (3) shartlarga ko‘ra

$$U_n(x,t) = \sum_{n=1}^{+\infty} \varphi_n e^{-\gamma t} E_\alpha(-\lambda_n t^\alpha)\sin(\pi n x) \qquad (13)$$

ko‘rinishda topamiz [ ].

Teorema: Agar $\varphi(x) \in C^2[0,1] \cap C^3(0,1)$, $\varphi'''(x) \in L_2(0,1)$, $\varphi(0) = 0$, $\varphi(1) = 0$, $\varphi''(0) = 0$, $\varphi''(1) = 0$ bo‘lsa, u holda

$$U_n(x,t) = \sum_{n=1}^{+\infty} \varphi_n e^{-\gamma t} E_\alpha(-\lambda_n t^\alpha)\sin(\pi n x) \qquad (14)$$

funksiya masalaning yechimini aniqlaydi.

Isbot. (14) qatorning yig‘indisi ham tenglamani qanoatlantiradi, hamda (14) qator tekis yaqinlashuvchi va birinchi hamda ikkinchi tartibli differensiallari mavjud bo‘lsa, (1) tenglamani qanoatlantiradi.

Dastlab qatorning tekis yaqinlashuvchiligini tekshiramiz.

$$|U(x,t)| = \left|\sum_{n=1}^{+\infty} \varphi_n e^{-\gamma t} E_\alpha(-\lambda t^\alpha)\sin(\pi n x)\right| \le$$

$$\le \sum_{n=1}^{+\infty} \left|\varphi_n e^{-\gamma t} E_\alpha(-\lambda t^\alpha)\sin(\pi n x)\right| \qquad (15)$$

tengsizlik o‘rinli bo‘ladi. Shuningdek $\left|e^{-\gamma t}\right| < 1$ va $\left|E_\alpha(-\lambda_n t^\alpha)\right| < C$ tengsizliklar o‘rinli ekanligidan foydalanamiz.

$$\sum_{n=1}^{+\infty}\left|\varphi_n e^{-\gamma t}E_\alpha(-\lambda t^\alpha)\sin(\pi n x)\right| \le C\sum_{n=1}^{+\infty}|\varphi_n| = C\sum_{n=1}^{+\infty}|\varphi_n| n\frac{1}{n} \le C\left[\sum_{n=1}^{+\infty}\frac{1}{n^2}\sum_{n=1}^{+\infty}n^2\varphi_n^2\right]^{\frac{1}{2}}$$

(16)

(16) ga Koshi-Bunyakovskiy tengsizligini qo'llaymiz.

$$\varphi_n = \sqrt{2}\int_0^1 \varphi(x)\sin(\pi n x)dx \qquad (17)$$

$$n\varphi_n = \sqrt{2}n\int_0^1 \varphi(x)\sin(\pi n x)dx \qquad (18)$$

(18) tenglikni bo'laklab integrallaymiz

$$n\varphi_n = \sqrt{2}n\int_0^1 \varphi(x)\sin(\pi n x)dx =$$

$$= \sqrt{2}n\varphi(x)\left(-\frac{1}{\pi n}\right)\cos(\pi n x)\Big|_0^1 + \sqrt{2}n\int_0^1\frac{1}{\pi n}\cos(\pi n x)\varphi'(x)dx = \frac{\sqrt{2}}{\pi}\int_0^1\varphi'(x)\cos(\pi n x)dx$$

$$n\varphi_n = \frac{\sqrt{2}}{\pi}\int_0^1\varphi'(x)\cos(\pi n x)dx \qquad (19)$$

(19) ga Bessel tengsizligini qo'llaymiz

$$\sum_{n=1}^{+\infty}n^2\varphi_n^2 \le \frac{2}{\pi^2}\int_0^1[\varphi'(x)]^2dx \qquad (20)$$

$\varphi(x)\in C^2[0,1]$, $\varphi'''(x)\in L_2(0,1)$.

(20) tengsizlikka ko'ra (14) absolyut va tekis yaqinlashuvchi bo'ladi.

(14) yechimni x bo'yicha birinchi va ikkinchi tartibli differensiallarini hisoblaymiz.

$$U_x(x,t) = \sum_{n=1}^{+\infty}(\pi n)\varphi_n e^{-\gamma t}E_\alpha(-\lambda_n t^\alpha)\cos(\pi n x)$$

$$U_{xx}(x,t) = -\sum_{n=1}^{+\infty}(\pi n)^2\varphi_n e^{-\gamma t}E_\alpha(-\lambda_n t^\alpha)\sin(\pi n x). \qquad (21)$$

Endi (21) tenglikni baholaymiz.

$$|U_{xx}(x,t)| = \left|-\sum_{n=1}^{+\infty}(\pi n)^2\varphi_n e^{-\gamma t}E_\alpha(-\lambda_n t^\alpha)\sin(\pi n x)\right| \le$$

$$\le \sum_{n=1}^{+\infty}\left|-(\pi n)^2\varphi_n e^{-\gamma t}E_\alpha(-\lambda_n t^\alpha)\sin(\pi n x)\right|$$

$|e^{-\gamma t}| < 1$ va $|E_\alpha(-\lambda t^\alpha)| < C$ tengsizliklar o'rinli ekanligidan foydalanib,

$$\sum_{n=1}^{+\infty}\left|-(\pi n)^2\varphi_n e^{-\gamma t}E_\alpha(-\lambda_n t^\alpha)\sin(\pi n x)\right| < C\pi^2\sum_{n=1}^{+\infty}|\varphi_n| n^2 =$$

$$= Cp^2 \sum_{n=1}^{+\infty} |\varphi_n| n^2 n \frac{1}{n} \le Cp^2 \left(\sum_{n=1}^{+\infty} \varphi_n^2 n^6 \sum_{n=1}^{+\infty} \frac{1}{n^2} \right)^{\frac{1}{2}},$$

$$Cp^2 \left(\sum_{n=1}^{+\infty} \varphi_n^2 n^6 \sum_{n=1}^{+\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} \qquad (22)$$

(22) ni hosil qilamiz. (22) da $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ yaqinlashuvchi qator hisoblanadi, u holda $\sum_{n=1}^{+\infty} \varphi_n^2 n^6$ qator yaqinlashuvchi ekanligini ko‘rsatamiz, buning uchun (17) tenglikni n^3 ga ko‘paytirib

$$n^3 \varphi_n = \sqrt{2} n^3 \int_0^1 \varphi(x) \sin(\pi n x) dx \qquad (23)$$

(23) ni hosil qilamiz va uni uch marta ketma-ket bo‘laklab integrallaymiz

$$\sqrt{2} n^3 \int_0^1 \varphi(x) \sin(\pi n x) dx = \sqrt{2} n^3 \varphi(x) \left(\frac{1}{-\pi n} \right) \cos(\pi n x) \Big|_0^1 + \frac{\sqrt{2} n^2}{\pi} \int_0^1 \varphi'(x) \cos(\pi n x) dx =$$

$$= \frac{\sqrt{2} n^2}{\pi} \int_0^1 \varphi'(x) \cos(\pi n x) dx,$$

$$\frac{\sqrt{2} n^2}{\pi} \int_0^1 \varphi'$$

$$-\frac{\sqrt{2} n}{\pi^2} \int_0^1 \varphi''(x) \sin(\pi n x) dx = \frac{\sqrt{2}}{\pi} \varphi''(x) \cos(\pi n x) \Big|_0^1 - \frac{\sqrt{2}}{\pi^2} \int_0^1 \varphi'''(x) \cos(\pi n x) dx = \quad ,$$

$$= -\frac{\sqrt{2}}{\pi^2} \int_0^1 \varphi'''(x) \cos(\pi n x) dx.$$

U holda (23) quyidagi ko‘rinishda bo‘ladi,

$$n^3 \varphi_n = -\frac{\sqrt{2}}{\pi^2} \int_0^1 \varphi'''(x) \cos(\pi n x) dx \qquad (24)$$

(24) ga bo‘lsa Bessel tengsizligini qo‘llaymiz:

$$\sum_{n=1}^{+\infty} n^6 \varphi_n^2 \le \int_0^1 \left[\frac{\sqrt{2}}{\pi} \varphi'''(x) \right]^2 dx. \qquad (25)$$

(25) tengsizlikka ko‘ra $U_{xx}(x,t)$ absolyut va tekis yaqinlashuvchi bo‘ladi. (1) tenglamadan ${}^{TC}D_{0t}^{\alpha,\gamma} U(x,t)$ ham absolyut va tekis yaqinlashuvchi bo‘ladi.

Teorema isbotlandi.

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