A PROBLEM FOR A THREE-DIMENSIONAL EQUATION OF MIXED TYPE WITH SINGULAR COEFFICIENT

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INTRODUCTION

Problem Statement

The study of boundary value problems for mixed-type equations is one of the central problems of the theory of partial differential equations of its appliedimportance. For the first time, F.I. Frankl [1] found important applications of these problems in gas dynamics, and I.N. Vekua [2] pointed out the importance of the problem of mixed-type equations in solving problems arising in the momentless theory of shells.

So far, the studies of boundary value problems for mixed-type equations with singular coefficients have been carried out mainly in the case of two independent variables. However, such problems in three-dimensional domains remain poorly studied.

The Tricomi problem for a mixed elliptic-hyperbolic equation in three-dimensional space using the method of integral Fourier transform was first studied in [3]. After this work, a number of works appeared in which boundary value problems for various elliptic-huperbolic equations in three-dimensional domains were considered (see, for example, [4], [5], [6], [7], [8], [9], [10], [11], [12]).

В данной работе изучается пространственная задача Трикоми для трехмерного уравнения смешанного типа с сингулярным коэффициентом в области, эллиптическая часть которой четверти цилиндра, а гиперболическая часть – треугольная прямая призма.

Let $\Omega=\{(x,y,z):(x,y)\in\Delta, z\in(0,c)\}$ where Δ - is the finite one-connected region of the plane xOy , bounded at $y\ge 0$ by the arc $\bar{\sigma}_0=\left\{(x,y):x^2+y^2=1, x\ge 0, y\ge 0\right\}$ and segment $\overline{OM}=\left\{(x,y):x=0, 0\le y\le 1\right\}$, and at $y\le 0$ - by segments $\overline{OQ}=\left\{(x,y):x+y=0, 0\le x\le 1/2\right\}$ and, $\overline{QP}=\left\{(x,y):x-y=1, 1/2\le x\le 1\right\}$ $O=O(0,0)$ $M=M(0,1)$ $P=P(1,0)$ $Q=Q(1/2,-1/2)$.

Let introduce the notations: $\Omega_0=\Omega\cap(y>0)$ $\Omega_1=\Omega\cap(y<0)$;

$\Delta_0=\Delta\cap(y>0),;,\Delta_1=\Delta\cap(y<0)$ $S_0=\left\{(x,y,z):\sigma_0\times(0,c)\right\}$,

$S_1=\left\{(x,y,z):OM\times(0,c)\right\}$, ;$\bar{S}_2=\left\{(x,y,z):\overline{OQ}\times[0,c]\right\}$;

$\bar{S}_3=\left\{(x,y,z):\bar{\Omega}\cap(z=0)\right\}$, $\bar{S}_4=\left\{(x,y,z):\bar{\Omega}\cap(z=c)\right\}$

In the domain Ω consider the equation

$$U_{xx} + (\mathrm{sgny})U_{yy} + U_{zz} + \frac{2\gamma}{z}U_z = 0, \qquad (1)$$

Where is. $\gamma \in (0,1/2)$

In the region Ω equation (1) belongs to a mixed type, namely in the region Ω_0 elliptic type, and in the region Ω_1- hyperbolic type, and $z=0$ are the planes of singularity of the equation, and when passing through the rectangle $\bar{\Omega}_0 \cap \bar{\Omega}_1$ the equation changes its type.

We investigate the following problem for equation (1) in the region Ω.

Problem T (Trikomi). Find the function $U(x,y,z)$, satisfying in the region Ω equation (1) and the following conditions:

$$U(x,y,z) \in C(\bar{\Omega}) \cap C^{2,2,2}_{x,y,z}(\Omega_0 \cup \Omega_1),\ U_x, U_y, z^{2\gamma}U_z \in C(\bar{\Omega}_0); \qquad (2)$$

$$U(x,y,z)\big|_{S_0} = F(x,y,z); \qquad (3)$$

$$U(x,y,z)\big|_{S_1} = 0,\ U(x,y,z)\big|_{\bar{S}_2} = 0\ , \qquad (4)$$

$$U(x,y,z)\big|_{\bar{S}_3} = 0,\ U(x,y,z)\big|_{\bar{S}_4} = 0\ , \qquad (5)$$

as well as the bonding condition

$$U_y(x,-0,z) = U_y(x,+0,z), \quad x \in (0,1),\ z \in (0,c), \qquad (6)$$

where $F(x,y,z)$ is a given function.

Note that the posed problem at $\gamma = 0$ is studied in [13].

2. Construction of partial solutions of equation (1) in the region of hyperbolicity and ellipticity of the equation

We find nontrivial solutions of equation (1) satisfying conditions (4), (5). Dividing the variables by the formula $U(x,y,z) = w(x,y)Z(z)$, from equation (1) and boundary conditions (4) and (5), we obtain the following problems:

$$w_{xx} + (\mathrm{sgn}\, y)w_{yy} - \lambda w = 0,\ \ (x,y) \in \Delta \cap \{x > 0\}, \qquad (7)$$

$$w(0,y) = 0,\ \ y \in (0,1);\quad w(x,-x) = 0,\ \ x \in [0,1/2]; \qquad (8)$$

$$Z''(z) + \frac{2\gamma}{z}Z'(z) + \lambda Z(z) = 0,\ 0 < z < c;\ Z(0) = 0,\ Z(c) = 0. \qquad (9)$$

Problem (9) has nontrivial solutions of the form [14], [15], [16]

$$Z_m(z) = z^{1/2-\gamma} J_{1/2-\gamma}(\sigma_m z/c),\ m \in N, \qquad (10)$$

where $J_l(z)-$ is the Bessel function [17], and $\sigma_m - m$ is the positive root of the equation , $J_{1/2-\gamma}\left(\sqrt{\lambda}c\right) = 0\ \lambda_m = (\sigma_m/c)^2\ m \in N$

According to [17], the system of eigenfunctions (10) is orthogonal and complete in space $L_2(0,c)$ with weight. $z^{2\gamma}$

Now consider the problem $\{(7),(8)\}$ at $\lambda=\lambda_m$ in the region Δ_1, i.e., consider the following problem:

$$w_{xx}-w_{yy}-\lambda_m w=0,\quad (x,y)\in\Delta_1, \tag{11}$$

$$w(x,-x)=0,\quad x\in[0,1/2]. \tag{12}$$

The solution of this problem is found in the form

$$w(x,y)=X(\xi)Y(\eta),\ \text{where}\ \xi=\sqrt{x^2-y^2},\ \eta=x^2/\xi^2 \tag{13}$$

Then, with respect to the functions $X(\xi)$ and $Y(\eta)$ we obtain the following conditions , $X(0)=0\left|\lim\limits_{\eta\to+\infty}Y(\eta)\right|<+\infty$ and equations

$$\xi^2X''(\xi)+\xi X'(\xi)-\left[\lambda_m\xi^2+\mu\right]X(\xi)=0,\ \xi>0; \tag{14}$$

$$\eta(1-\eta)Y''(\eta)+[1/2-\eta]Y'(\eta)+\frac{1}{4}\mu Y(\eta)=0,\ \eta>1, \tag{15}$$

where $\mu\in R-$ is the separation .

Solutions of equation (14) satisfying the condition $X(0)=0$, exist at $\mu>0$ and they (with accuracy to a constant multiplier) are of the form [17]

$$X(\xi)=I_\omega(\sigma_m\xi/c),\ m\in N, \tag{16}$$

Where $\omega=\sqrt{\mu}$, and $I_l(x)$ is the Bessel function of an imaginary argument of order l [17].

(15) is a hypergeometric Gaussian equation [18]. Its general solution is defined by the formula [18]

$$Y(\eta)=c_1\eta^{-\omega/2}F(\omega/2,1/2+\omega/2,1+\omega;1/\eta)+$$

$$c_2\eta^{\omega/2}F(-\omega/2,1-\omega/2,1-\omega;1/\eta), \tag{17}$$

where c_1,c_2 –are arbitrary constants.

$\omega>0$ it sincefollows from (17) that in order to obtain the function bounded at $\eta\to+\infty$, we need to put $c_2=0$ in the formula , as a result of which, we get

$$Y(\eta)=c_1\eta^{-\omega/2}F(\omega/2,1/2+\omega/2,1+\omega;1/\eta). \tag{18}$$

Consequently, continuous and nontrivial in $\overline{\Delta}_1$ solution of the problem $\{(11),(12)\}$, according to (13), (16) and (18), are defined by the equations

$$w_m^-(x,y)=c_1\eta^{-\omega/2}F(\omega/2,(1+\omega)/2,1+\omega;1/\eta)I_\omega(\sigma_m\xi/c),\qquad c_1\neq0,\ m\in N.$$

(19)

Hence we find

$$\begin{cases} \tau_m^-(x) = \lim\limits_{y\to -0} w_m^-(x,y) = c_1 2^{\omega} I_{\omega}(\sigma_m x/c), \ x\in[0,1]; \\ \nu_m^-(x) = \lim\limits_{y\to -0} \dfrac{\partial}{\partial y} w_m^-(x,y) = c_1 \omega 2^{\omega} x^{-1} I_{\omega}(\sigma_m x/c), \ x\in(0,1), \end{cases} \quad (20)$$

Where $\Gamma(z)-$ is the Euler gamma function [18].

Now consider the problem $\{(7),(8)\}$ at $\lambda=\lambda_m$ in the region Δ_0, i.e., consider the following problem:

$$w_{xx}+w_{yy}-\lambda_m w=0, \quad (x,y)\in\Delta_0, \quad (21)$$

$$w(0,y)=0, \quad y\in(0,1). \quad (22)$$

Dividing the variables by formula

$$w(x,y)=Q(\rho)S(\varphi), \quad (23)$$

where $\rho=\sqrt{x^2+y^2}$, $\varphi=arctg(y/x)$, from equation (21) and conditions $w\in C(\overline{\Delta}_0)$, (22), we obtain the following problems:

$$\rho^2 Q''(\rho)+\rho Q'(\rho)-\left[(\sigma_m\rho/c)^2+\tilde{\mu}\right]Q(\rho)=0, \ \rho\in(0,1), \quad (24)$$

$$|Q(0)|<+\infty; \quad (25)$$

$$S''(\varphi)+\tilde{\mu}S(\varphi)=0, \ \varphi\in(0,\pi/2), \quad (26)$$

$$S(\pi/2)=0, \quad (27)$$

where $\tilde{\mu}\in R$ is the separation constant.

We first study the problems $\{(24),(25)\}$. The general solution of equation (24) is defined in the form [17]

$$Q_m(\rho)=c_3 I_{\tilde{\omega}}(\sigma_m\rho/c)+c_4 K_{\tilde{\omega}}(\sigma_m\rho/c), \ \rho\in[0,1], \quad (28)$$

here , $\tilde{\omega}=\sqrt{\tilde{\mu}}$ c_3 and c_4 are arbitrary constants, $K_l(x)$is a Macdonald function of order l [17]

It follows from (28) that solutions of equation (24), satisfying condition (25), exist $\tilde{\mu}\geq 0$ at and they are defined by equations

$$Q_m(\rho)=c_3 I_{\tilde{\omega}}(\sigma_m\rho/c), \ \tilde{\omega}\geq 0, \ m\in N. \quad (29)$$

Now, let us study the problem $\{(26),(27)\}$. The general solution of equation (26) is

$$S(\varphi)=c_5\cos(\tilde{\omega}\varphi)+c_6\sin(\tilde{\omega}\varphi), \quad (30)$$

where . c_5 and c_6 are arbitrary constants.

Satisfying function (30) with condition (27), we obtain $c_6=k_3(\tilde{\omega})c_5$, where $k_3(\tilde{\omega})=-ctg(\tilde{\omega}\pi/2)$. Substituting $c_6=k_3(\tilde{\omega})c_5$ into (30) and assuming $c_5=1$ (this does not violate generality), we have

$$S(\varphi)=\cos(\tilde{\omega}\varphi)-ctg(\tilde{\omega}\pi/2)\sin(\tilde{\omega}\varphi). \quad (31)$$

Based on (23), (29) and (31), we conclude that the continuous and nontrivial in $\bar{\Delta}_0$ solution of the problem $\{(21),(22)\}$, have the form

$$w_m^+(x,y) = c_3 I_{\tilde{\omega}}(\sigma_m \rho / c)\left[\cos(\tilde{\omega}\varphi) - ctg(\tilde{\omega}\pi/2)\sin(\tilde{\omega}\varphi)\right], \quad c_3 \neq 0,\ m \in N \quad (32)$$

Hence, by direct calculation, we find

$$\begin{cases} \tau_m^+(x) = \lim\limits_{y\to+0} w_m^+(x,y) = c_3 I_{\tilde{\omega}}(\sigma_m x/c), \ x \in [0,1]; \\ \nu_m^+(x) = \lim\limits_{y\to+0} \dfrac{\partial}{\partial y} w_m^+(x,y) = -c_3\tilde{\omega} ctg(\tilde{\omega}\pi/2) x^{-1} I_{\tilde{\omega}}(\sigma_m x/c), \ x \in (0,1). \end{cases} \quad (33)$$

Then ,based on $U(x,y,z) = w(x,y)Z(z)$ and the notation introduced, the following equations follow from the conditions and $U(x,y,z) \in C(\bar{\Omega})$ and (6):

$$\begin{cases} \tau_m^-(x) = \tau_m^+(x), \ x \in [0,1], \\ \nu_m^-(x) = \nu_m^+(x), \ x \in (0,1). \end{cases} \quad (34)$$

Substituting (20) and (33) into (34) and assuming $\omega = \tilde{\omega}$, we have a homogeneous system of equations with respect to c_1 and c_3:

$$\begin{cases} 2^{\omega} c_1 + ctg\dfrac{\omega\pi}{2} c_3 = 0, \\ 2^{\omega} c_1 - c_3 = 0. \end{cases} \quad (35)$$

From system (35), we find $ctg\dfrac{\omega\pi}{2} = -1$. Writing out the solutions of this equation and taking into account the condition $\omega > 0$ we find

$$\omega_n = 2n - 1/2, \ n \in N. \quad (36)$$

Based on (36), the numbers $\mu_n = \omega_n^2, n \in N$ are the eigenvalues of problems $\{(15), \left|\lim\limits_{\eta\to+\infty} Y(\eta)\right| < +\infty\}$ and $\{(26),(27)\}$.

Note that at $\omega = \omega_n$ the function $S(\varphi)$, defined by the equality (31), will be written in the form

$$S_n(\varphi) = \sqrt{2}\sin\left[\left(2n - \frac{1}{2}\right)\varphi + \frac{\pi}{4}\right]. \quad (37)$$

In [19], it was proved that the system of eigenfunctions (37) forms a basis in the space $L_2(0,\pi/2)$.

Taking into account the above proven and equality (19), (32), $\omega = \tilde{\omega} = \omega_n$, we conclude that the functions

$$w_{nm}(x,y)=\begin{cases} c_3\sqrt{2}\sin\left[\left(2n-\frac{1}{2}\right)\varphi+\frac{\pi}{4}\right]I_{2n-\frac{1}{2}}(\sigma_m\rho/c),\ (x,y)\in\bar{\Delta}_0, \\ c_3 2^{-\omega_n}\left(\frac{1}{\eta}\right)^{\omega_n/2}F\left(n-\frac{1}{4},n+\frac{1}{4},2n+\frac{1}{2};\frac{1}{\eta}\right)I_{2n-\frac{1}{2}}\left(\frac{\sigma_m\xi}{c}\right),(x,y)\in\bar{\Delta}_1, \end{cases}$$

(38)

are continuous and nontrivial in $\bar{\Delta}$ solving of the problem $\{(7),(8)\}$.

Then, the functions

$$U_{nm}(x,y,z)=w_{nm}(x,y)Z_m(z),\ n,m\in N, \qquad (39)$$

where $Z_m(z)$ and $w_{nm}(x,y)$ are the functions defined by equalities (10) and (38) are continuous and nontrivial in Ω solutions of equation (1) satisfying conditions (4)-(5).

3. Singularity of the solution of the problem T

Let $U(x,y,z)=V(\rho,\varphi,z)-$ solve the problem T in the domain Ω_0 and satisfy the condition

$$V_\varphi(\rho,0,z)=\omega_n V(\rho,0,z), \qquad (40)$$

where $\rho,\varphi,z-$ are the cylindrical coordinates, realted to Cartesian coordinates by the equations , $\rho=\sqrt{x^2+y^2}$ $\varphi=arctg(y/x)$ $z=z$.

In these coordinates, equations (1) and condition (3) are written in the form

$$V_{\rho\rho}+\frac{1}{\rho^2}V_{\varphi\varphi}+\frac{1}{\rho}V_\rho+V_{zz}+\frac{2\gamma}{z}V_z=0,\ (\rho,\varphi,z)\in\tilde{\Omega}; \qquad (41)$$

$$V(1,\varphi,z)=f(\varphi,z),\ \varphi\in[0,\pi/2],\ z\in[0,c], \qquad (42)$$

where $\tilde{\Omega}=\{(\rho,\varphi,z):\rho\in(0,1),\ \varphi\in(0,\pi/2),\ z\in(0,c)\}$

$f(\varphi,z)=F(\cos\varphi,\sin\varphi,z)$.

Using $V(\rho,\varphi,z)$ and eigenfunctions (10), (37), let us compose the following function:

$$\zeta_{nm}(\rho)=d_m\int_0^c\int_0^{\pi/2}V(\rho,\varphi,z)S_n(\varphi)z^{2\gamma}Z_m(z)d\varphi dz,\ n,m\in N, \qquad (43)$$

Where $d_m=2\Big/\left[cJ_{3/2-\gamma}(\sigma_m)\right]^2$.

Based on (43), we introduce the functions

$$\zeta_{nm}^{\varepsilon_1\varepsilon_2}(\rho)=d_m\int_{\varepsilon_2}^{c-\varepsilon_2}\int_{\varepsilon_1}^{\pi/2-\varepsilon_1}V(\rho,\varphi,z)S_n(\varphi)z^{2\gamma}Z_m(z)d\varphi dz, \qquad (44)$$

where ε_1 and ε_2 are sufficiently small positive numbers.

Obviously, $\lim\limits_{\varepsilon_1,\varepsilon_2 \to 0} \zeta_{nm}^{\varepsilon_1\varepsilon_2}(\rho) = \zeta_{nm}(\rho)$.

From (44), we find $\left(\dfrac{\partial^2}{\partial\rho} + \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\right)\zeta_{nm}^{\varepsilon_5\varepsilon_6}(\rho)$:

$$\left(\frac{\partial^2}{\partial\rho} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\zeta_{nm}^{\varepsilon_1\varepsilon_2}(\rho) = d_m \int_{\varepsilon_1}^{c-\varepsilon_1}\int_{\varepsilon_2}^{\pi/2-\varepsilon_2}\left(\frac{\partial^2}{\partial\rho} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)V(\rho,\varphi,z)S_n(\varphi)z^{2\gamma}Z_m(z)\,d\varphi dz$$

.

Taking into account equations (41), from the latter we have

$$\left(\frac{\partial^2}{\partial\rho} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\zeta_{nm}^{\varepsilon_1\varepsilon_2}(\rho) = -\frac{d_m}{\rho^2}\int_{\varepsilon_2}^{c-\varepsilon_2}\left[\int_{\varepsilon_1}^{\pi/2-\varepsilon_1} V_{\varphi\varphi}S_n(\varphi)\,d\varphi\right]z^{2\gamma}Z_m(z)\,dz -$$

$$-d_m\int_{\varepsilon_1}^{\pi/2-\varepsilon_1}\left[\int_{\varepsilon_2}^{c-\varepsilon_2}\left(V_{zz} + \frac{2\gamma}{z}V_z\right)z^{2\gamma}Z_m(z)\,dz\right]S_n(\varphi)\,d\varphi.$$

Applying the rule integration by parts from the last one, we obtain

$$\left(\frac{\partial^2}{\partial\rho} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\zeta_{nm}^{\varepsilon_1\varepsilon_2}(\rho) = -\frac{d_m}{\rho^2}\int_{\varepsilon_2}^{c-\varepsilon_2}\left\{\left\{\left[V_\varphi S_n(\varphi) - VS_n'(\varphi)\right]\right\}\Big|_{\varphi=\varepsilon_1}^{\varphi=\pi/2-\varepsilon_1} -\right.$$

$$\left. -\mu_n\int_{\varepsilon_1}^{\pi/2-\varepsilon_1} V(\rho,\varphi,z)S_n(\varphi)d\varphi\right\}z^{2\gamma}Z_m(z)\,dz -$$

$$-d_m\int_{\varepsilon_1}^{\pi/2-\varepsilon_1}\left\{\left\{\left[V_z(\rho,\varphi,z)Z_m(z) - V(\rho,\varphi,z)Z_m'(z)\right]z^{2\gamma}\right\}\Big|_{z=\varepsilon_2}^{z=c-\varepsilon_2} -\right.$$

$$\left. -(\sigma_{\gamma m}/c)^2\int_{\varepsilon_2}^{c-\varepsilon_2} V(\rho,\varphi,z)z^{2\gamma}Z_m(z)\,dz\right\}S_n(\varphi)\,d\varphi. \qquad (45)$$

Hence, passing to the limit at , $\varepsilon_1 \to 0$ $\varepsilon_2 \to 0$ and considering (2), (4), (5), (27), (40) and boundary conditions of the problems (9), as well as the notation (43), we obtain the equality

$$\zeta_{nm}''(\rho) + \frac{4\beta+1}{\rho}\zeta_{nm}'(\rho) - \left(\lambda_m - \frac{\mu_n}{\rho^2}\right)\zeta_{nm}(\rho) = 0,\ \rho \in (0,1).$$

Hence ,the function $\zeta_{nm}(\rho)$ satisfies the differential equation (24) at . $\mu = \mu_n$.

Moreover ,due to the boundary conditions (3), it follows from (43) that the function $\zeta_{nm}(\rho)$ satisfies the following boundary conditions:

$$\zeta_{nm}(1) = f_{nm}, \qquad (46)$$

where

$$f_{nm} = d_m \int_0^c \int_0^{\pi/2} f(\varphi, z) S_n(\varphi) z^{2\gamma} Z_m(z) d\varphi dz. \qquad (47)$$

Consequently, the function $\zeta_{nm}(\rho)$, defined by equality (43), satisfies equation (24) at $\tilde{\mu} = \mu_n$ and conditions (25), (46). Therefore,by subjecting the general solution (28) of equation (24) to these conditions, we find the coefficients c_3 and c_4 : $c_3 = f_{nm} / I_{\omega_n}(\sigma_m / c)$, $c_4 = 0$.

Substituting these values in (28), we unambiguously find the function $\zeta_{nm}(\rho)$

$$\zeta_{nm}(\rho) = I_{\omega_n}(\sigma_m \rho / c) f_{nm} / I_{\omega_n}(\sigma_m / c). \qquad (48)$$

Now we can prove the following theorem.

Theorem 1: If there exists a solution to the problem T when condition (40) is satisfied, then it is singular.

Proof. For this, purpose it suffices to prove that the homogeneous problem T, has only a trivial solution. Let $f(\varphi, z) \equiv 0$ .Then $f_{nm} = 0$ for all $n, m \in N$. By virtue of this equality, it follows from (48) and (43) that $\int_0^c \int_0^{\pi/2} V(\rho, \varphi, z) S_n(\varphi) z^{2\gamma} Z_m(z) d\varphi dz = 0$.

Hence, by virtue of the complenteness of the system of functions (10) with weight $z^{2\gamma}$ in the space $L_2(0, c)$ and $V(\rho, \varphi, z) \in C(\bar{\tilde{\Omega}})$ it follows that, $\int_0^{\pi/2} V(\rho, \varphi, z) S_n(\varphi) d\varphi = 0$ $n \in N$. Given the completeness of the system of functions (37) in the space $L_2(0, \pi/2)$ and $V(\rho, \varphi, z) \in C(\bar{\tilde{\Omega}})$, it follows from the last equality that $V(\rho, \varphi, z) \equiv 0$ in $\bar{\tilde{\Omega}}$.

Using this equality and $U(x, y, z) = V(\rho, \varphi, z)$, it is easy to see that $U(x, +0, z) \equiv 0,\ U_y(x, 0, z) \equiv 0,\ x \in [0,1],\ z \in [0, c]$.

Then, by virtue of $U(x, y, z) \in C(\bar{\Omega})$, the following equations are true

$$U(x, -0, z) \equiv 0,\ U_y(x, -0, z) \equiv 0,\ x \in [0,1],\ z \in [0, c]. \qquad (49)$$

It follows from the results of [20] that the solution of Eq .

$$U_{xx} - U_{yy} + U_{zz} + \frac{2\gamma}{z} U_z = 0,\ (x, y, z) \in \Omega_1$$

satisfying conditions (49) is identically zero, i.e., $U(x, y, z) \equiv 0,\ (x, y, z) \in \bar{\Omega}_1$. Theorem 1 is proved.

4. Construction and justification of the solution of the problem T

Substituting the values $c_3 = f_{nm} / I_{\omega_n}\left(\sigma_m / c\right)$ to equality (38), and then the obtained function in (39), we find partial solutions of the problem T in the form of

$$U_{nm}(x,y,z) = \begin{cases} U_{nm}^{+}(x,y,z), \ (x,y,z) \in \bar{\Omega}_0, \ n,m \in N, \\ U_{nm}^{-}(x,y,z), \ (x,y,z) \in \bar{\Omega}_1, \ n,m \in N, \end{cases}$$

where

$$U_{nm}^{+}(x,y,z) = Z_m(z)\zeta_{nm}(\rho)S_n(\varphi), \ (x,y,z) \in \bar{\Omega}_0, \quad (50)$$

$$U_{nm}^{-}(x,y,z) = 2^{-\omega_n} Z_m(z) X_{nm}(\xi) Y_n(\eta), \ (x,y,z) \in \bar{\Omega}_1, \quad (51)$$

$$X_{nm}(\xi) = I_{2n-1/2}(\sigma_m \xi / c) f_{nm} / I_{2n-1/2}(\sigma_m / c), \ \xi = \sqrt{x^2 - y^2}, \quad (52)$$

$$Y_n(\eta) = (1/\eta)^{n-1/4} F(n-1/4, n+1/4, 2n+1/2; 1/\eta), \ \eta = x^2 / \xi^2, \quad (53)$$

and, $Z_m(z)$ $S_n(\varphi)$ f_{nm} and $\zeta_{nm}(\rho)$ are determined by the equations (10), (37), (47) and (48) respectively.

Theorem 2. If $f(\varphi, z)$ satisfies the following conditions:

I. $f(\varphi, z) \in C_{\varphi,z}^{4,5}(\bar{\Pi})$, where $\Pi = \{(\varphi, z) : \varphi \in (0, \pi/2), z \in (0, c)\}$;

II. $\left.\dfrac{\partial^j}{\partial \varphi^j} f(\varphi, z)\right|_{\varphi=0} = 0$, $\left.\dfrac{\partial^j}{\partial \varphi^j} f(\varphi, z)\right|_{\varphi=\pi/2} = 0$ $j = \overline{0,3}$;

III. $\left.\dfrac{\partial^j}{\partial z^j} f(\varphi, z)\right|_{z=0} = 0$, $\left.\dfrac{\partial^j}{\partial z^j} f(\varphi, z)\right|_{z=c} = 0$ $j = \overline{0,4}$.

Then the solution of the problem T exists and is determined by the formula

$$U(x,y,z) = \begin{cases} \sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty} U_{nm}^{+}(x,y,z), \ (x,y,z) \in \bar{\Omega}_0, \\ \sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty} U_{nm}^{-}(x,y,z), \ (x,y,z) \in \bar{\Omega}_1, \end{cases} \quad (54)$$

where , $U_{nm}^{+}(x,y,z)$ $U_{nm}^{-}(x,y,z)$ are functions defined by formulas (50) and (51).

Before proceeding to the proof of this theorem, let us prove some lemmas.

Lemma 1. If $\gamma \in (0, 1/2)$, then the following estimates are valid with respect to the functions $Z_m(z)$, defined by equations (10), at $z \in [0, c]$ and sufficiently large m:

$$|Z_m(z)| \le c_5 z^{1-2\gamma} (\sigma_m)^{1/2-\gamma}, \quad (55)$$

$$|z^{2\gamma} Z_m'(z)| \le c_6 (\sigma_m)^{1/2} \quad (56)$$

$$|B_{\gamma-1/2}^{z} Z_m(z)| \le c_7 z^{1-2\gamma} (\sigma_m)^{5/2-\gamma}, \quad (57)$$

where $c_j,\ j=\overline{5,7}-$ are positive constants, $B_q^y \equiv \dfrac{\partial^2}{\partial y^2}+\dfrac{2q+1}{y}\dfrac{\partial}{\partial y}$ –Bessel operator [21].

Proof: Let us rewrite the function $Z_m(z)$ in the from

$$Z_m(z)=\frac{(2c)^{\gamma-1/2}}{\Gamma(3/2-\gamma)}z^{1-2\gamma}(\sigma_m)^{1/2-\gamma}\,\bar{J}_{1/2-\gamma}(\sigma_m z/c), \qquad (58)$$

where $\bar{J}_\nu(z)-$is the Bessel-Clifford function [22]:

$$\bar{J}_\nu(z)=\Gamma(\nu+1)(z/2)^{-\nu}J_\nu(z)=\sum_{j=0}^{\infty}\frac{\left(-z^2/4\right)^j}{(\nu+1)_j\,j!}.$$

The function $\bar{J}_\nu(z)$ is even and infinitely differentiable. Moreover,the equality $\bar{J}_\nu(0)=1$ the inequality $\left|\bar{J}_\nu(z)\right|\le 1$ at $\nu>-1/2$ are valid.Considering this and $1/2-\alpha>0$, from equality (58), we get an estimate (55).

Now, consider the function $z^{2\gamma}Z'_m(z)=\dfrac{\sigma_m}{c}z^{1/2+\gamma}J_{-1/2-\gamma}(\sigma_m z/c)$. Let us rewrite this function in the form

$$z^{2\gamma}Z'_m(z)=(\sigma_m/c)^{1/2-\gamma}\,\xi^{1/2+\gamma}J_{-1/2-\gamma}(\xi), \qquad (59)$$

where $\xi=\sigma_m z/c$. The function $\xi^{1/2+\gamma}J_{-1/2-\gamma}(\xi)$ is bounded at the point $\xi=0$ and continuous at $\xi\in[0,+\infty)$. Moreover, by virtue of the asymptotic formula of the Bessel function:

$$J_\nu(\xi)\approx\left(\frac{2}{\pi\xi}\right)^{1/2}\cos\left(\xi-\frac{\nu\pi}{2}-\frac{\pi}{4}\right), \qquad (60)$$

for sufficiently large ξ, the estimate $\left|\xi^{1/2+\gamma}J_{-1/2-\gamma}(\xi)\right|<\xi^{\gamma}\tilde{c}_6$, where $\tilde{c}_6=const>0$ is valid. Considering these properties of the function $\xi^{1/2+\gamma}J_{-1/2-\gamma}(\xi)$, it follows from (60) that for sufficiently large ξ the inequality $\left|z^{2\gamma}Z'_m(z)\right|\le\tilde{c}_6(\sigma_m/c)^{1/2-\gamma}\,\xi^{\gamma}=\tilde{c}_6(\sigma_m/c)^{1/2}z^{\gamma}\le c_6(\sigma_m)^{1/2}$, i.e., the estimate (56) is valid.

It is known that the function $Z_m(z)$ satisfies equation from (9) at $\lambda_m=(\sigma_m/c)^2$. It follows that $B^z_{\gamma-1/2}Z_m(z)=-(\sigma_m/c)^2 Z_m(z)$. Then, by virtue of evaluation (55), evaluation (57) is valid. Lemma 1 is proven.

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