SINGULYAR KOEFFITSIYENTLI TENGLAMA UCHUN YARIM POLOSADA NOLOKAL MASALA

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Annotatsiya. *Ushbu maqolada singulyar koeffitsiyentli aralash parabolik tenglama uchun nolokal shartli masala qo'yilgan va masala yechimining mavjudligi va yagonaligi isbotlangan.*

$Q=\{(x,t):-l\le x,\ 0<t<T\}$ yarim polosada quyidagi

$$0=Lu\equiv\begin{cases}L_1^{(k)}u\equiv u_{xx}+\frac{k}{x}u_x-u_t\ (x,t)\in Q_1=Q\cap(x>0),\\ L_2u\equiv u_{tt}+u_x, \qquad (x,t)\in Q_2=Q\cap(x<0)\end{cases}$$

aralash parabolik tenglamani qaraylik, bu yerda $T,l,k\in R$ bo'lib, $T>0$, $l>0$, $k\in(-1;1)$.

$Lu=0$ - Q sohada aralash parabolik tenglama bo'lib, Q_1 sohada to'g'ri parabolik, Q_2 sohada esa teskari parabolikdir. Aralash parabolik tenglamalar uchun chegaraviy masalalar birinchi bo'lib fransuz matematigi Mario Jevre [1] tomonidan o'rganilgan. Keyinchalik, tadqiqotchilar tomonidan aralash parabolik tenglamalar uchun lokal va nolokal masalalar o'rganishga bo'lgan qiziqish ortdi. Jumladan, [2] ishda vaqt yo'nalishlari almashinuvchi aralash parabolik tenglama uchun Jevre masalasi tadqiq qilingan bo'lsa, [3] ishda aralash parabolik tenglama uchun turli lokal va nolokal shartli masalalar qo'yilgan va o'rganilgan. A.M.Naxushevning [4] ishi esa xarakteristik formalari o'zgaruvchi ikkinchi tartibli parabolik tenglamalar uchun korrekt masalalar qo'yish va masalalar korrektligi uchun zaruriy va yetarli shartlar aniqlashga bag'ishlangan.

Dastlab, tadqiqotchilar tomonidan ikkinchi tartibli aralash parabolik tenglamalar qaralgan bo'lsa, keyinchalik yuqori tartibli tenglamalar uchun masalalar tadqiqoti T.D.Djuraev, S.A.Tersenev, D.Amanov, S.V. Popov va ularning shogirdlari tomonidan rivojlantirildi.

So'ngi vaqtlarda tadqiqotchilar tomonidan kasr tartibli differensial operatorlarni o'z ichiga oluvchi aralash parabolik tenglamalar uchun ham tadqiqotlar olib borilmoqda. Jumladan, [5] ishda ikkinchi tartibli aralash parabolik tenglama uchun Jevre masalasi o'rganilgan bo'lsa, [6] ishda to'rtinchi tartibli aralash parabolik tenglama uchun masalalar spektral analiz usuli bilan tadqiq qilingan.

Yuqoridagi ishlarda qaralayotgan tenglamalarning vaqt yo'nalishlari kollinear bo'lib, vaqt yo'nalishlari kollinear bo'lmagan aralash parabolik tenglamalar uchun masalalar kam o'rganilgan. [7] ishda vaqt yo'nalishlari perpendikulyar bo'lgan aralash parabolik tenglama uchun nolokal shartli masalalar tadqiq qilingan bo'lsa, [8],[9] ishlarda kasr tartibli differensial operatorlarni o'z ichiga oluvchi aralash parabolik tenglamalar uchun turli nolokal shartli masalalar o'rganilgan.

$\tilde{L}u=0$ tenglama uchun Q sohada ushbu masalani o'rganamiz.

***I* masala.** Q sohaning yopig'ida aniqlangan va uzluksiz shunday $u(x,t)$ funksiya topilsinki, u Q_1 va Q_2 sohalarda mos ravishda $L_1^{(k)}u=0$ va $L_2u=0$ tenglamalarning regulyar yechimi bo'lib, quyidagi ulash shartini

$$\lim_{x\to-0}u_x(x,t)=\lim_{x\to+0}x^k u_x(x,t),\quad 0<t<T \qquad (1)$$

va ushbu

$$u(x,0)=\varphi_1(x),\ 0\le x<+\infty \qquad (2)$$

$$\lim_{x\to+\infty}u(x,t)=0,\ \ t\in[0;T] \qquad (3)$$

$$u_t(x,0)=\varphi_3(x),\quad -l\le x\le 0 \qquad (4)$$

$$u(x,T)=a(x)\int_0^T u(x,t)dt+\varphi_2(x),\quad -l\le x\le 0 \qquad (5)$$

chegaraviy shartlarni qanoatlantirsin, bu yerda $\varphi_j(x)$, $j=\overline{1,3}-$o'zining aniqlanish sohasida uzluksiz bo'lgan berilgan funksiyalar bo'lib, $\varphi_1(0)=0$, $\lim_{x\to+\infty}\varphi_1(x)=0$

Masala yechimining mavjudligini va yagonaligini o'rganamiz. Faraz qilaylik, $u(x,t)$-qo'yilgan masalaning yechimi bo'lsin. Masala shartlariga asoslanib,

$$u(+0,t)=u(-0,t)=\tau(t) \qquad (6)$$

$$\lim_{x\to+0}x^k u_x(x,t)=\nu(t),\qquad 0<t<T \qquad (7)$$

belgilashlarni qabul qilaylik.

Ma'lumki, $L_1^{(k)}u=0$ tenglamaning Q_1 sohaning yopig'ida aniqlangan va uzluksiz hamda (2), (3) va (7) shartlarni qanoatlantiuvchi yechimi quyidagi ko'rinishda aniqlanadi [11]:

$$u(x,t)=\int_0^{+\infty}\frac{\xi^k(x\xi)^{(1-k)/2}}{2t}I_{(k-1)/2}\left(\frac{x\xi}{2t}\right)e^{-(x^2+\xi)/4t}\varphi_1(\xi)d\xi-$$

$$-2^{-k}\Gamma^{-1}\left(\frac{1+k}{2}\right)\int_0^t\nu(\eta)(t-\eta)^{-(1+k)/2}e^{-x^2/4(t-\eta)}d\eta. \qquad (8)$$

(8) formulada $x\to+0$ da limitga o'tib va (6) belgilashni e'tiborga olib, noma'lum $\tau(t)$va $\nu(t)$ funksiyalar orasidagi quyidagi funksional munosabatga ega bo'lamiz:

$$\tau(t)=\Phi(t)-2^{-k}\Gamma^{-1}\left(\frac{1+k}{2}\right)\int_0^t\nu(\eta)(t-\eta)^{-(1+k)/2}d\eta,$$

bu yerda

$$\Phi(t)=2^{-k}\Gamma^{-1}\left(\frac{1+k}{2}\right)t^{-(1+k)/2}\int_0^{+\infty}\xi^k e^{-\xi^2/(4t)}\varphi_1(\xi)d\xi.$$

Agar $\tau(t)$ funksiyani vaqtincha ma'lum funksiya deb hisoblasak, oxirgi tenglik $\nu(t)$ noma'lum funksiyaga nisbatan Abel integral tenglamasi bo'ladi. Bunday tenglamaning yechim formulasidan [12] foydalanib, $\nu(t)$ funksiyani topamiz:

$$\nu(t)=2^{k}\Gamma^{-1}\left(\frac{1-k}{2}\right)\frac{d}{dt}\int_{0}^{t}(t-\eta)^{\frac{k-1}{2}}\left[\Phi(\eta)-\tau(\eta)\right]d\eta. \qquad (9)$$

Endi masala shartlarini va (8), (6) belgilashlarni e'tiborga olib, $L_2u=0$ tenglama va (2), (5) shartlarda x ni nolga intiltiramiz:

$$\tau''(t)+\lim_{x\to -0}u_x(x,t)=0, \quad 0<t<T, \qquad (10)$$

$$\tau(0)=0, \quad \tau(T)=\tau(h)+\varphi_2(0). \qquad (11)$$

(1) ulash shartini, (6) belgilashni va (9) tenglikni e'tiborga olsak, (10) tenglikdan

$$\tau''(t)-2^{k}\Gamma^{-1}\left(\frac{1-k}{2}\right)\frac{d}{dt}\int_{0}^{t}(t-\eta)^{(k-1)/2}\tau(\eta)d\eta=\Phi_1(t), \quad 0<t<T \qquad (12)$$

tenglamaga ega bo'lamiz, bu yerda

$$\Phi_1(t)=-2^{k}\Gamma^{-1}\left(\frac{1-k}{2}\right)\frac{d}{dt}\int_{0}^{t}(t-\eta)^{(k-1)/2}\Phi(\eta)d\eta.$$

Agar (12) tenglamaning (11) shartlarni qanoatlantiruvchi $\tau(t)$ yechimini topsak, $\nu(t)$ funksiya (9) formula bilan topiladi. Unda qo'yilgan masalaning yechimi Q_1 sohada (7) formula bilan aniqlanadi. Shuning uchun {(12), (11)} masalani yechish bilan shug'ullanamiz.

Shu maqsadda (12) formulada t ni z bilan almashtirib, z bo'yicha $[0,t]$ oraliqda ketma-ket ikki marta integrallaymiz. Natijada, $\tau'(0)=C$ belgilash kiritib, $\tau(0)=0$ ekanligini e'tiborga olib, $\tau(t)$ noma'lum funksiyaga nisbatan quyidagi

$$\tau(t)-\frac{2^{k+1}}{k+1}\Gamma^{-1}\left(\frac{1-k}{2}\right)\int_{0}^{t}\tau(\eta)(t-\eta)^{(1+k)/2}d\eta=Ct+\int_{0}^{t}\Phi_1(\eta)(t-\eta)d\eta, \quad 0\le t\le T \qquad (13)$$

integral tenglamaga ega bo'lamiz.

(13) – ikkinchi tur Volterra integtal tenglamasi bo'lib, uning yagona yechimi ushbu formula bilan aniqlanadi [12]:

$$\tau(t)=\frac{d}{dt}\int_{0}^{t}E_{(k+3)/2,1}\left[\lambda(t-\eta)^{\frac{k+3}{2}}\right]\left[C\eta+\Phi_2(\eta)\right]d\eta, \qquad (14)$$

bu yerda $E_{\alpha,\beta}(z)$- Mittag-Leffler funksiyasi [12]:

$$\lambda=2^{k}\Gamma\left(\frac{1+k}{2}\right)\Gamma^{-1}\left(\frac{1-k}{2}\right), \qquad \Phi_2(t)=\int_{0}^{t}\Phi_1(\eta)(t-\eta)d\eta.$$

(14) formulada differensiallash amalini bajarib, so'ngra

$$\frac{d}{dt}E_{(k+3)/2,1}\left[\lambda(t-\eta)^{(k+3)/2}\right]=-\frac{d}{d\eta}E_{(k+3)/2,1}\left[\lambda(t-\eta)^{(k+3)/2}\right]$$

formuladan foydalanib va bo'laklab integrallash qoidasini qo'llab hamda

$$\Phi_2(0)=0, \qquad E_{(k+3)/2,1}(0)=1,$$

$$\int_{0}^{t}E_{(k+3)/2,1}\left[\lambda(t-\eta)^{(k+3)/2}\right]d\eta=tE_{(k+3)/2,2}\left[\lambda t^{(k+3)/2}\right]$$

tengliklarni e'tiborga olib, $\tau(t)$ni quyidagicha topamiz:

$$\tau(t)=CtE_{(k+3)/2,2}\left[\lambda t^{(k+3/2)}\right]+\int_0^t E_{(k+3)/2,1}\left[\lambda(t-\eta)^{(k+3)/2}\right]\Phi_2'(\eta)d\eta. \qquad (15)$$

(15) formuladan yordamida $\tau(T)$ va $\int_0^T \tau(t)dt$ larni hisoblab, so'ngra ularni (11) shartga qo'yib, C noma'lumni bir qiymatli aniqlaymiz:

$$CT\left(E_{(k+3)/2,2}\left[\lambda(t-\eta)^{(k+3)/2}\right]-a(0)T\cdot E_{(k+3)/2,3}\left[\lambda T^{(k+3)/2}\right]\right)=$$

$$=a(0)\int_0^T E_{(k+3)/2,2}\left[\lambda(t-\eta)^{(k+3)/2}\right](T-\eta)\Phi_2'(\eta)d\eta-\int_0^T E_{(k+3)/2,1}\left[\lambda(T-\eta)^{(k+3)/2}\right]\Phi_2'(\eta)d\eta \quad (16)$$

Agar $a(0)$ va T sonlar uchun quyidagi tengsizlik

$$E_{(k+3)/2,2}\left[\lambda T^{(k+3)/2}\right]-a(0)\cdot T\cdot E_{(k+3)/2,3}\left[\lambda T^{(k+3)/2}\right]\neq 0 \qquad (17)$$

bajarilgan bo'lsa, (15) tenglikdan C noma'lum son bir qiymatli topiladi.

1-izoh. Masalan, $a(0)T\leq 2$ bo'lganda (17) tengsizlik bajariladi.

C noma'lumning (16) tenglikdan topilgan qiymatini (15) tenglikka qo'ysak, {(12),(11)} masalaning yechimi bo'lgan $\tau(t)$ funksiyani to'lig'icha aniqlaymiz.

$\tau(t)$ funksiya topilgandan so'ng $\nu(t)$ funksiya (9) tenglik bilan aniqlanadi. Shundan so'ng 3.6-maslaning yechimi D_1 sohada (8) formula bilan topiladi, D_2 sohada esa $L_2u=0$ tenlamaning (4), (5) va $u(0,t)=\tau(t)$, $0\leq t\leq T$ shartlarni qanoatlantiruvchi yechimi sifatida topiladi. Oxirgi masalani I_0 deb belgilaymiz va bir qiymatli yechilishini isbotlaymiz.

Faraz qilaylik, $u(x,t)-I_0$ masalaning yechimi bo'lsin. $u(x,T)=\varphi(x)$, $-l\leq x\leq 0$ belgilash kiritaylik. U holda, $u(x,t)$ funksiyani D_2 sohada $L_2u=0$ tenglama uchun birinchi chegaraviy maslaning yechimi sifatida

$$u(x,t)=\int_0^T \tau(\eta)G(0,\eta;x,t)d\eta+\int_x^0 \varphi_2(\xi)G_\eta(\xi,0;x,t)d\xi-$$

$$-\int_x^0 \varphi(\xi)G_\eta(\xi,T;x,t)d\xi-\int_x^0 \varphi(\xi)G_\eta(\xi,T;x,t)d\xi \qquad (18)$$

ko'rinishida yozish mumkin bo'ladi, bu yerda

$$G(\xi,\eta;x,t)=\frac{1}{2\sqrt{\pi(\xi-x)}}\times\sum_{n=-\infty}^{\infty}\left\{\exp\left[-\frac{(y-\eta-2n)^2}{4(\xi-x)}\right]-\exp\left[-\frac{(y+\eta-2n)^2}{4(\xi-x)}\right]\right\},\xi>x$$

.

$u(x,T)=\varphi(x)$ belgilashni va (18) formulani e'tiborga olib, (5) shartdan

$$\varphi(x)+\int_x^0 \varphi(\xi)\left[a(x)K(x,\xi)\right]d\xi=f_1(x), \quad -l\leq x\leq 0 \qquad (19)$$

integral tenglamaga ega bo'lamiz, bu yerda

$$K(x,\xi)=-\frac{1}{2\sqrt{\pi(\xi-x)}}\sum_{n=-\infty}^{\infty}\left[e^{\frac{-n^2}{\xi-x}}-e^{\frac{-(2n+T)^2}{4(\xi-x)}}+e^{\frac{-(2n-T)^2}{4(\xi-x)}}\right],\ \xi>x.$$

$$f_1(x)=a(x)\int_0^T\left[\int_0^1\tau(\eta)G(0,\eta;x,t)d\eta+\int_x^0\varphi_2(\xi)G(\xi,0;x,t)d\xi\right]dy+\varphi_3(x).$$

(19) – Volterraning ikkinchi tur integral tenglamasidir. Uning yadrosi (1/2) tartibli sust maxsuslikka ega bo'lib, $x\to\xi$ da $(x-\xi)^{-1/2}$ funksiya kabi tutadi, o'ng tomoni esa $C[-l,0]$.

Shuning uchun (19) integral tenglama $[-l,0]$ oraliqda uzluksiz bo'lgan yagona yechimga ega. Demak, I_0 masala ham yagona yechimga ega.

Shunday qilib, quyidagi teorema o'rinli ekanligi isbotlandi.

Teorema. Agar $a(0)\neq 0$, $x\in[-l;0]$ bo'lib, $a(0)$ va T sonlar (17) tengsizlikni qanoatlantirsa, masala yagona yechimga ega bo'ladi.

2-izoh. $a(x)$, $x\in[-l;0]$ bo'lganda ham *I*-masalaning yechimi D_2 sohada (18) formula bilan aniqlanadi, faqat bunda $\varphi(x)=\varphi_3(x)$ deb olinadi.

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