RIMAN-LIUVILL MA'NOSIDAGI KASR TARTIBLI DIFFERENSIAL OPERATORNI O'Z ICHIGA OLUVCHI CHIZIQLI DIFFERENSIAL TENGLAMALAR SISTEMASINI YECHISHNING DALAMBER USULI

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Annotatsiya: *Ushbu ishda Riman-Liuvill ma'nosidagi kasr tartibli differensial operatorni o'z ichiga oluvchi chiziqli differensial tenglamalar sistemasi uchun Koshi masalasi yechimi Dalamber usulidan foydalanib aniqlangan.*

Quyidagi

$$\begin{cases} D_{0t}^{\alpha}x(t) = ax + by + f_1(t), \\ D_{0t}^{\alpha}y(t) = cx + dy + f_2(t) \end{cases} \qquad (1)$$

sistemaning

$$\lim_{t\to+0} I_{0t}^{1-\alpha}x(t) = x_0, \quad \lim_{t\to+0} I_{0t}^{1-\alpha}y(t) = y_0 \qquad (2)$$

shartlarni qanoatlantiruvchi yechimini topish masalasini qaraylik, bu yerda D_{0t}^{α} va $I_{0t}^{1-\alpha}$ lar mos holda kasr tartibli Riman-Liuvill ma'nosidagi differensial va integral operatorlar [1]:

$$D_{at}^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-z)^{-\alpha} f(z)dz,$$

$$I_{0t}^{1-\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-z)^{-\alpha} f(z)dz,$$

$\Gamma(z)$-Eylerning gamma funksiyasi, $\alpha, a, b, c, d, x_0, y_0$,-berilgan haqiqiy sonlar bo'lib, $0<\alpha<1$; $f_1(t)$ va $f_2(t)$- berilgan funksiyalar, $x(t)$ va $y(t)$lar esa noma'lum funksiyalar.

$\{(1),(2)\}$ Koshi masalasining yechimini topish bilan shug'ullanamiz. Shu maqsadda (1) ning ikkinchi tenglamasini λ songa ko'paytirib so'ngra birinchi tenglamaga qo'shamiz. Natijada, yig'indini differensiallash formulasini e'tiborga olgan holda ushbu

$$D_{0t}^{\alpha}[x(t)+\lambda y(t)] = (a+\lambda c)x + (b+\lambda d)y + f_1(t) + \lambda f_2(t) \qquad (3)$$

tenglikni hosil qilamiz.

(3) tenglikning o'ng tomonidagi birinchi hadni qavsdan tashqariga chiqarib

$$D_{0t}^{\alpha}[x(t)+\lambda y(t)] = (a+\lambda c)\left(x + \frac{b+\lambda d}{a+\lambda c}y\right) + f_1(t) + \lambda f_2(t) \qquad (4)$$

tenglikni hosil qilamiz.

λ sonni shunday tanlaylikki, u ushbu $\dfrac{b+\lambda d}{a+\lambda c}=\lambda$ tenglamaning yechimi bo'lsin. Faraz qilaylik, λ_1 va λ_2 oxirgi tenglamaning ildizlari bo'lib, $\lambda_1 \neq \lambda_2$ bo'lsin. U holda bu sonlarni (4) ga qo'yib, ushbu

$$D_{0t}^{\alpha}(x(t)+\lambda_i y(t))=(a+\lambda_i c)\left(x+\frac{b+\lambda_i d}{a+\lambda_i c}y\right)+f_1(t)+\lambda_i f_2(t), \qquad i=1,2 \tag{5}$$

ko'rinishdagi $x(t)+\lambda_i y(t)$ noma'lumlarga nisbatan chiziqli ikkita tenglamaga ega bo'lamiz. Soddalik maqsadida quyidagi belgilashlarni kiritaylik:

$$x(t)+\lambda y(t)=z(t), \quad f_1(t)+\lambda_i f_2(t)=f_{3,i}(t), \quad a+\lambda_i c=\tilde{\lambda}_i, \quad i=1,2. \tag{6}$$

U holda (5) tenglama ushbu ko'rinishga keladi:

$$D_{0t}^{\alpha}z(t)=\tilde{\lambda}_i z(t)+f_{3,i}(t), \qquad i=1,2. \tag{7}$$

Ushbu

$$I_{0t}^{\alpha}D_{0t}^{\alpha}z(t)=z(t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}I_{0t}^{1-\alpha}z(a) \tag{8}$$

formulani e'tiborga olgan holda (7) tenglikning har ikki tomoniga I_{0t}^{α} operatorni tatbiq qilamiz. Natijada (2) shartlarni e'tiborga olsak, $z(t)$ noma'lum funksiyaga nisbatan

$$z(t)-\tilde{\lambda}_i\int_0^t (t-\eta)^{\alpha-1}z(\eta)d\eta=I_{0t}^{\alpha}f_{3,i}(t)+\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+\lambda_i y_0) \tag{9}$$

ko'rinishdagi 2-tur Volterra integral tenglamasiga ega bo'lamiz. Soddalik maqsadida quyidagicha belgilash kiritaylik:

$$I_{0t}^{\alpha}f_{3,i}(t)+\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+\lambda_i y_0)=f_{4,i}(t). \tag{10}$$

U holda (9) tenglama ushbu ko'rinishga keladi:

$$z(t)-\tilde{\lambda}_i\int_0^t (t-\eta)^{\alpha-1}z(\eta)d\eta=f_{4,i}(t). \tag{11}$$

Oxirgi tenglama yechimini ketma-ket yaqinlashish usulidan foydalanib topamiz. Nolinchi yaqinlashish sifatida $f_{4,i}(t)$ ni qabul qilamiz.

$$z_0(t)=f_{4,i}(t).$$

Birinchi va ikkinchi yaqinlashishlarni mos holda quyidagi formulalar orqali aniqlaymiz:

$$z_1(t)=f_{4,i}(t)+\tilde{\lambda}_i\int_0^t K(t,\eta)f_{4,i}(\eta)d\eta,$$

$$z_2(t)=f_{4,i}(t)+\tilde{\lambda}_i\int_0^t K(t,\tau)\left[f_{4,i}(\tau)+\tilde{\lambda}_i\int_0^n K(\tau,\eta)f_{4,i}(\eta)d\eta\right]d\tau=$$

$$= f_{4,i}(t)+\tilde{\lambda}_i\int_0^t K(t,\eta)f_{4,i}\eta d\eta+\tilde{\lambda}_i^2\int_o^t K_2(t,\eta)\,f_{4,i}(\eta)d\eta\,.$$

$n-$yaqinlashishni esa

$$z_n = f_{4,i}(t)+\int_0^t\sum_{j=0}^n K_j(t,\eta)\,f_{4,i}(\eta)d\eta$$

formuladan foydalanib topamiz, bu yerda $K_j(t,\eta)$-iteratsiyalangan yadrolar bo'lib,

$$K_0(t,\eta)=K(t,\eta),\quad K_j(t,\eta)=\int_\eta^t K(t,\tau)K_{j-1}(\tau,\eta)d\tau.$$

Dastlab, $K_2(t,\eta)$ni hisoblaylik:

$$K_2(t,\eta)=\int_\eta^t K(t,\tau)K(\tau,\eta)d\tau=\int_\eta^t\frac{(t-\tau)^{\alpha-1}(\tau-m)^{\alpha-1}}{\Gamma^2(\alpha)}d\tau\,.$$

Ushbu almashtirishni bajaramiz: $(t-\eta)s+\eta=\tau$. U holda beta va gamma funksiya xossalaridan foydalansak,

$$K_2(t,\eta)=\int_0^1\frac{(t-\eta)^{2\alpha-1}(1-s)^{\alpha-1}s^{\alpha-1}}{\Gamma^2(\alpha)}ds=\frac{(t-\eta)^{2\alpha-1}}{\Gamma^2(\alpha)}\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}ds=$$

$$=\frac{(t-\eta)^{2\alpha-1}}{\Gamma^2(\alpha)}B(\alpha,\alpha)=\frac{(t-\eta)^{2\alpha-1}\Gamma^2(\alpha)}{\Gamma^2(\alpha)\Gamma(2\alpha)}=\frac{(t-\eta)^{2\alpha-1}}{\Gamma(2\alpha)}.$$

Matematik induksiya usulidan foydalanib, ko'rsatish mumkinki, $K_j(t,\eta)$uchun quyidagi ifoda o'rinli:

$$K_j(t,\eta)=\frac{(t-\eta)^{j\alpha-1}}{\Gamma(j\alpha)}.$$

Endi $R(t,\eta;\tilde{\lambda}_i)$ rezolventani tuzamiz:

$$R(t,\eta;\tilde{\lambda}_i)=\sum_{j=1}^{\infty}\tilde{\lambda}_i^{j-1}K_j(t,\eta)=\sum_{j=1}^{\infty}\frac{\tilde{\lambda}_i^{j-1}(t-\eta)^{j\alpha-1}}{\Gamma(j\alpha)}=\sum_{j=0}^{\infty}\frac{\tilde{\lambda}_i^{j}(t-\eta)^{j\alpha+\alpha-1}}{\Gamma(j\alpha+\alpha)}.$$

$E_{\alpha,\beta}(z)$-Mittag-Leffler funksiyasining yoyilmasidan foydalansak, rezolventani quyidagicha yozishimiz mumkin:

$$R(t,\eta;\tilde{\lambda}_i)=(t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[\tilde{\lambda}_i(t-\eta)^{\alpha}\right].$$

U holda 2-tur Volterra integral tenglamalari nazariyasiga asosan (11) tenglama yechimi ushbu formula orqali aniqlanadi:

$$z(t)=f_{4,i}(t)+\tilde{\lambda}_i\int_0^t(t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[\tilde{\lambda}_i(t-\eta)^{\alpha}\right]f_{4,i}(\eta)d\eta\,.$$

Endi oxirgi tenglikka asosan quyidagi tenglamalar sistemasini tuzamiz:

$$\begin{cases} x(t)+\lambda_1 y(t)= f_{4,1}(t)+\tilde{\lambda}_1\int\limits_0^t (t-\eta)^{\alpha-1} E_{\alpha,\alpha}\left[\tilde{\lambda}_1 (t-\eta)^{\alpha}\right] f_{4,1}(\eta)d\eta \\ x(t)+\lambda_2 y(t)= f_{4,2}(t)+\tilde{\lambda}_2\int\limits_0^t (t-\eta)^{\alpha-1} E_{\alpha,\alpha}\left[\tilde{\lambda}_2 (t-\eta)^{\alpha}\right] f_{4,2}(\eta)d\eta \end{cases}. \quad (12)$$

Bu sistemadan Koshi masalasining yechimi bir qiymatli aniqlanadi.

Misol. Koshi masalasi yechimini toping:

$$\begin{cases} D_{0t}^{\alpha} x(t) = -x-6y+t^2, \\ D_{0t}^{\alpha} y(t) = x+4y+t^3, \end{cases}$$

$$\lim_{t\to+0} I_{0t}^{1-\alpha} x(t) = x_0, \quad \lim_{t\to+0} I_{0t}^{1-\alpha} y(t) = y_0.$$

Yechish. Masala shartlariga asosan

$$a=-1, b=-6, c=1, d=4, f_1(t)=t^2, f_2(t)=t^3.$$

U holda (4) tenglama quyidagi ko'rinishni oladi:

$$D_{0t}^{\alpha}\left[x(t)+\lambda y(t)\right] = (-1+\lambda)\left(x+\frac{-6+4\lambda}{-1+\lambda}y\right)+t^2+\lambda t^3. \quad (13)$$

Bundan $\frac{-6+4\lambda}{-1+\lambda}=\lambda$ tenglamani yechib, $\lambda_1=2$ va $\lambda_2=3$ ildizlarni topamiz. Bu sonlarni (13) tenglamaga qo'yib, (12) formuladan foydalansak, $x(t)$ va $y(t)$ noma'lumlarga nisbatan quyidagi algebraik tenglamalar sistemasiga ega bo'lamiz:

$$\begin{cases} x(t)+2y(t)= f_{4,1}(t)+\int\limits_0^t (t-\eta)^{\alpha-1} E_{\alpha,\alpha}\left[(t-\eta)^{\alpha}\right] f_{4,1}(\eta)d\eta \\ x(t)+3y(t)= f_{4,2}(t)+2\int\limits_0^t (t-\eta)^{\alpha-1} E_{\alpha,\alpha}\left[2(t-\eta)^{\alpha}\right] f_{4,2}(\eta)d\eta. \end{cases} \quad (14)$$

Kasr tartibli integral ta'rifidan foydalanib, $f_{4,1}(t)$ va $f_{4,2}(t)$ funksiyalarni soddalashtiramiz. Dastlab, $f_{4,1}(t)$ ni qaraylik:

$$f_{4,1}(t)=\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-\xi)^{\alpha-1}\left(\xi^2+2\xi^3\right)d\xi+\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+2y_0)=$$

$$=\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-\xi)^{\alpha-1}\xi^2 d\xi+\frac{2}{\Gamma(\alpha)}\int\limits_0^t (t-\xi)^{\alpha-1}\xi^3 d\xi+\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+2y_0).$$

Integrallarda $\xi = st$ almashtirish bajaramiz. Natijada beta va gamma funksiya xossalaridan foydalanib, quyidagiga ega bo'lamiz:

$$f_{4,1}(t)=\frac{t^{\alpha-2}}{\Gamma(\alpha)}\int_0^1 s^2(1-s)^{\alpha-1}ds+\frac{2t^{\alpha-3}}{\Gamma(\alpha)}\int_0^1 s^3(1-s)^{\alpha-1}ds+\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+2y_0)=$$

$$=\frac{t^{\alpha-2}}{\Gamma(\alpha)}B(\alpha,1)+\frac{2t^{\alpha-3}}{\Gamma(\alpha)}B(\alpha,2)+\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+2y_0)=$$

$$=\frac{t^{\alpha-2}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha-3}}{\Gamma(\alpha+2)}+\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+2y_0).$$

Yuqoridagi kabi ko'rsatish mumkinki,

$$f_{4,2}(t)=\frac{t^{\alpha-2}}{\Gamma(\alpha+1)}+\frac{t^{\alpha-3}}{\Gamma(\alpha+2)}+\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+3y_0).$$

$f_{4,1}(t)$ va $f_{4,2}(t)$ funksiyalarning olingan ifodalarini e'tiborga olsak, (14) quyidagi ko'rinishni oladi:

$$x(t)+2y(t)=\frac{t^{\alpha-2}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha-3}}{\Gamma(\alpha+2)}+\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+2y_0)+$$

$$+\int_0^t(t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[(t-\eta)^{\alpha}\right]f_{4,1}(\eta)d\eta;$$

$$x(t)+3y(t)=\frac{t^{\alpha-2}}{\Gamma(\alpha+1)}+\frac{t^{\alpha-3}}{\Gamma(\alpha+2)}+\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+3y_0)+$$

$$+2\int_0^t(t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[2(t-\eta)^{\alpha}\right]f_{4,2}(\eta)d\eta.$$

Oxirgi tengliklardan $x(t)$ va $y(t)$ noma'lum funksiyalar quyidagi ko'rinishda aniqlanadi:

$$\begin{cases} x(t)=\dfrac{t^{\alpha-2}}{\Gamma(\alpha+1)}-4\displaystyle\int_0^t(t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[2(t-\eta)^{\alpha}\right]f_{4,2}(\eta)d\eta+ \\ +3\displaystyle\int_0^t(t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[(t-\eta)^{\alpha}\right]f_{4,1}(\eta)d\eta+\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}x_0; \\ y(t)=\dfrac{t^{\alpha-3}}{\Gamma(\alpha+2)}+2\displaystyle\int_0^t(t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[2(t-\eta)^{\alpha}\right]f_{4,2}(\eta)d\eta- \\ -\displaystyle\int_0^t(t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[(t-\eta)^{\alpha}\right]f_{4,1}(\eta)d\eta+\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}y_0. \end{cases}$$