UCHINCHI TARTIBLI PARABOLO-GIPERBOLIK TENGLAMA UCHUN BIR NOLOKAL MASALANING YECHILISHI HAQIDA

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Annotatsiya. Ushbu maqola aralash tipdagi uchinchi tartibli parabolo-giperbolik tenglama uchun bayon qilingan bir nolokal masalaning bir qiymatli yechilishi tadqiqiga bag’ishlangan.

Kalit so‘zlar: parabolo-giperbolik tenglama, nolokal masala, integral tenglama, Volterra tenglamasi.

Ushbu

$$\frac{\partial}{\partial x}\left[u_{xx}-\frac{1}{2}(1-signy)u_{yy}-\frac{1}{2}(1-signy)u_{y}\right]=0 \qquad (1)$$

uchinchi tartibli tenglamani $D=D_1\cup I\cup D_2$ sohada qaraylik, bu yerda $D_1=\{(x,y):0<x<1,0<y\le 1\}$, $D_2=\{(x,y):0<x<1,(-y)<x<y+1\}$, $I=\{(x,0):0<x<1\}$.

D_1 sohada $y>0$ va $signy=1$ ekanligidan (1) tenglama

$$\frac{\partial}{\partial x}\left[u_{xx}-u_{y}\right]=0 \qquad (2)$$

ko‘rinishda yozilib, parabolik tipga tegishli bo‘ladi. (2) tenglamani x bo‘yicha integrallab,

$$u_{xx}-u_{y}=\omega_1(y),\ (x,y)\in D_1 \qquad (3)$$

ko‘rinishda yozish mumkin, bu yerda $\omega_1(y)$ – noma’lum funksiya.

D_2 sohada $y<0$ va $signy=-1$ bo‘lib, (1) tenglama giperbolik tipga tegishli va

$$\frac{\partial}{\partial x}\left[u_{xx}-u_{yy}\right]=0 \qquad (4)$$

ko‘rinishga ega bo’ladi. (4) tenglamani x bo‘yicha integrallasak,

$$u_{xx}-u_{yy}=\omega_2(y),\ (x,y)\in D_2 \qquad (5)$$

tenglamaga ega bo‘lamiz, bu yerda $\omega_2(y)$ – noma’lum funksiya.

D sohada (1) tenglama uchun quyidagi masalani qaraylik.

1-masala. Shunday $u(x,y)\in C(\overline{D})\cap C^{3,1}_{x,y}(D_1)\cap C^{3,2}_{x,y}(D_2)$ funksiya topilsinki, u D_1 va D_2 sohalarda (1) tenglamani hamda ushbu

$$u(0,y)=\varphi_1(y),\ u(1,y)=\alpha(y)u(0,y)+\varphi_2(y),\ 0\le y\le 1; \qquad (6)$$

$$u_x(x,y)\big|_{x=0}=\varphi_3(y),\qquad 0\le y\le 1; \qquad (7)$$

$$a(x)\frac{d}{dx}u\left(\frac{x}{2},-\frac{x}{2}\right)+b(x)\frac{d}{dx}u\left(\frac{x+1}{2},\frac{x-1}{2}\right)=c(x),\ 0\le x\le 1; \qquad (8)$$

$$\left.\frac{\partial u}{\partial n}\right|_{y=-x}=\psi(x), \qquad 0\le x\le\frac{1}{2}; \qquad (9)$$

$$u_y(x,+0)=u_y(x,-0), \qquad 0<x<1 \qquad (10)$$

shartlarni qanoatlantirsin, bu yerda $a(x),\ b(x),\ c(x),\ \psi(x),$ $\alpha(y),\ \varphi_1(y),\ \varphi_2(y),\ \varphi_3(y)$ – berilgan uzluksiz funksiyalar, n – $y=-x$ to‘g’ri chiziqqa o‘tkazilgan ichki normal.

Masalani tadqiq qilish uchun (10) shartga asosan quyidagi belgilashlarni kiritaylik:

$$u(x,0)=\tau(x), 0\le x\le 1;\ u_y(x,0)=\nu(x),\ 0<x<1;$$

$$\tau(x)\in C[0,1]\cap C^2(0,1), \nu(x)\in C(0,1)\cap L(0,1).$$

Qo‘yilgan masalani D_2 sohada qaraylik. Ma’lumki, yuqoridagi belgilashlarni e’tiborga olsak, D_2 sohada (5) tenglama uchun Koshi masalasining yechimi

$$u(x,y)=\frac{1}{2}\left[\tau(x+y)+\tau(x-y)\right]+\frac{1}{2}\int_{x-y}^{x+y}\nu(t)dt+\frac{1}{2}\int_0^y\int_{x-(y-t)}^{x+(y-t)}\omega_2(t)d\xi dt$$

(11)

formula bilan ifodalanadi. (11) formulani (9) shartga bo‘ysundirib, ba’zi hisoblashlarni bajarsak, ushbu

$$\omega_2(y)=-\sqrt{2}\psi'(-y), \qquad 0\le y\le\frac{1}{2} \qquad (12)$$

munosabatga ega bo‘lamiz.

Endi (11) formula bilan ifodalangan $u(x,y)$ funksiyani (8) shartga bo’ysundirib, $\tau(x)$ va $\nu(x)$ noma’lum funksiyalar orasidagi munosabat

$$\left[a(x)+b(x)\right]\tau'(x)+\left[b(x)-a(x)\right]\nu(x)=2c(x)+$$

$$+a(x)\int_0^{x/2}\omega_2(t)dt-b(x)\int_0^{(x-1)/2}\omega_2(t)dt, \qquad 0<x<1 \qquad (13)$$

ekanligini topamiz.

(12) tenglikni (13) munosabatga qo‘ysak,

$$\left[a(x)+b(x)\right]\tau'(x)+\left[b(x)-a(x)\right]\nu(x)=$$

$$=2c(x)+a(x)\int_0^{x/2}\left(-\sqrt{2}\psi'(-t)\right)dt-b(x)\int_0^{(x-1)/2}\left(-\sqrt{2}\psi'(-t)\right)dt, \qquad 0<x<1$$

(14)

hosil bo‘ladi. So‘nggi tenglikdagi oxirgi ikki integralda $-t=z$ almashtirish hamda hisoblashlarni bajarsak,

$$\int_0^{x/2}\psi'(-t)dt=-\int_0^{x/2}\psi'(-t)d(-t)=\{-t=z\}=$$

$$-\int_0^{x/2}\psi'(z)d(z)=-\int_0^{x/2}d(\psi(z))=-\psi\left(\frac{x}{2}\right)+\psi(0),$$

$$\int_0^{x-1/2}\psi'(-y)dt=-\int_0^{x-1/2}\psi'(-y)d(-y)=\{-y=z\}=$$

$$-\int_0^{x/2}\psi'(z)d(z)=-\int_0^{x/2}d(\psi(z))=-\psi\left(\frac{x-1}{2}\right)+\psi(0)$$

kelib chiqadi. Yuqoridagi hisoblashlarni e’tiborga olib, (14) munosabatdan ushbu

$$\left[a(x)+b(x)\right]\tau'(x)+\left[b(x)-a(x)\right]\nu(x)=f(x) \qquad (15)$$

tenglik hosil bo‘ladi, bu yerda

$$f(x)=2c(x)-\sqrt{2}\left\{a(x)\left[-\psi\left(\frac{x}{2}\right)+\psi(0)\right]-b(x)\left[-\psi\left(\frac{x-1}{2}\right)+\psi(0)\right]\right\}.$$

D_1 sohada (3) tenglama va (6), (7) shartlarda $y\to+0$ da limitga o‘tib,

$$\tau''(x)-\nu(x)=\omega_1(0),\ 0<x<1; \qquad (16)$$

$$\tau(0)=\varphi_1(0),\ \tau(1)=\alpha(0)\varphi_1(0)+\varphi_2(0),\ \tau'(0)=\varphi_3(0) \qquad (17)$$

munosabatlarga ega bo‘lamiz.

(15) munosabatga ko‘ra (16) tenglamani

$$\tau''(x)+p(x)\tau'(x)=f_1(x) \qquad (18)$$

ko‘rinishda yozish mumkin, bu yerda

$$p(x)=\frac{a(x)+b(x)}{b(x)-a(x)},$$

$$f_1(x)=\frac{f(x)+\omega_1(0)\left[b(x)-a(x)\right]}{b(x)-a(x)}.$$

Natijada $\{(17),(18)\}$ masala hosil bo‘ladi. Bu masalani tadqiq qilishda (18) tenglamani

$$\left[p_1(x)\tau'(x)\right]'=f_2(x) \qquad (19)$$

ko’rinishda yozib olamiz, bu yerda

$$p_1(x)=e^{\int_0^x p(t)dt},\ f_2(x)=f_1(x)e^{\int_0^x p(t)dt}.$$

{(17),(19)}masalada

$$\tau(x)=z(x)+\left[\alpha(0)-1\right]\varphi_1(0)x+\varphi_2(0)x+\varphi_1(0) \qquad (20)$$

almashtirish bajarib,

$$\left[p_1(x)z'(x)\right]' = f_3(x), \quad (21)$$

$$z(0)=0, \quad z(1)=0 \quad (22)$$

masalaga ega bo'lamiz, bu yerda

$$f_3(x)=f_2(x)+\left[\alpha(0)-1\right]\varphi_1(0)p_1'(x)+\varphi_2(0)p_2'(x).$$

{(21),(22)} masalaning yechimini Grin funksiyasi usuli bilan Gilbert teoremasiga asosan

$$\tau(x)=\int_0^1 p_1(s)G(x,s)ds \quad (23)$$

ko'rinishda topamiz, bu yerda $G(x,s)$ - Grin funksiyasi:

$$G(x,s)=\begin{cases} \dfrac{\int_0^x \frac{dz}{p_1(x)}\int_1^s \frac{dz}{p_1(x)}}{\int_0^1 \frac{dz}{p_1(x)}}, & 0\le x\le s, \\ \dfrac{\int_0^s \frac{dz}{p_1(x)}\int_x^1 \frac{dz}{p(x)}}{\int_0^1 \frac{dz}{p_1(x)}}, & 0\le x\le 1. \end{cases} \quad (24)$$

(23) munosabatga ko'ra noma'lum $\nu(x)$ funksiyani (15) ifoda orqali topiladi.

Demak, $\tau(x)$ funksiya (23), $\nu(x)$ funksiya esa (15) formuladan topilgandan so'ng o'rganilayotgan 1-masalaning yechimini D_2 sohada (11) formula bilan, D_1 sohada esa (5) tenglama uchun quyidagi $1'$-masala yechimi ko'rinishda yoziladi:

$$u_{xx}-u_y=\omega_1(y), \quad (x,y)\in D_1; \quad (25)$$

$$u(x,0)=\tau(x),\ 0\le x\le 1;\ u(0,y)=\varphi_1(y),\ u(1,y)=\varphi_2(y),\ 0\le y\le 1;$$

(26)

$$u_x(x,y)\big|_{x=0}=\varphi_3(y), \quad 0\le y\le 1. \quad (27)$$

Agar $\omega_1(y)$ funksiyani vaqtincha ma'lum deb hisoblasak, (25) tenglamaning (26) shartlarni qanoatlantiruvchi yechimini $[10]$

$$u(x,y)=\int_0^y \varphi_1(\eta)G_{1\xi}(x,y;0,\eta)d\eta-\int_0^y \varphi_2(\eta)G_{1\xi}(x,y;1,\eta)d\eta+$$

$$+\int_0^1 \tau(\xi)G_1(x,y;\xi,0)d\xi-\int_0^y \omega_1(\eta)d\eta\int_0^1 G_1(x,y;\xi,\eta)d\xi,\ (x,y)\in D_1 \quad (28)$$

ko'rinishida ifodalash mumkin, bu yerda $G(x,y;\xi,\eta)$ – Grin funksiyasi,

$$G_j(x,y;\xi,\eta)= \quad (29)$$

$$=\frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{m=-\infty}^{+\infty}\left\{\exp\left[-\frac{(x-\xi+2m)^2}{4(y-\eta)}\right]+(-1)^j\exp\left[-\frac{(x+\xi+2m)^2}{4(y-\eta)}\right]\right\},\quad j=\overline{1,2}$$

.

Endi (28) munosabatni (27) shartga bo‘ysundirish maqsadida, uni x bo‘yicha differensiallasak,

$$u_x=\int_0^y\varphi_1(\eta)G_{1\xi x}(x,y;0,\eta)d\eta-\int_0^y\varphi_2(\eta)G_{1\xi x}(x,y;1,\eta)d\eta+$$

$$+\int_0^1\tau(\xi)G_{1x}(x,y;\xi,0)d\xi-\int_0^y\omega_1(\eta)d\eta\int_0^1G_{1x}(x,y;\xi,\eta)d\xi$$

hosil bo‘ladi. Oxirgi tenglikni $u_x(0,y)=\varphi_3(y)$ shartga bo‘ysundirib, ba‘zi amallarni bajarsak, $\omega_1(y)$ funksiyaga nisbatan ushbu

$$\int_0^y\omega_1(\eta)M(y,\eta)d\eta=g_3(y),\qquad 0\le y\le 1,\qquad (30)$$

birinchi tur Volterra integral tenglamasiga ega bo‘lamiz, bu yerda

$$g_3(y)=\int_0^y\varphi_1(\eta)d\eta\int_0^1G_{1\xi x}(x,y;0,\eta)dx-\int_0^y\varphi_2(\eta)d\eta\int_0^1G_{1\xi x}(x,y;1,\eta)dx+$$

$$+\int_0^1\tau(\xi)d\xi\int_0^1G_{1x}(x,y;\xi,0)dx-\varphi_3(y),\qquad (31)$$

$$M(y,\eta)=\int_0^1\int_0^1G_{1x}(x,y;\xi,\eta)d\xi dx.$$

Ma’lumki [10],

$$\lim_{\eta\to y}\int_0^1G_{1x}(x,y;\xi,\eta)d\xi=1.\qquad (32)$$

(32) tenglikni e’tiborga olib, (30) differensial tenglamani y bo‘yicha differensiallab,

$$\omega_1(y)+\int_0^y\omega_1(\eta)(\partial/\partial y)M(y,\eta)d\eta=g_3'(y),\ 0<y<1\qquad (33)$$

ikkinchi tur Volterra integral tenglamasiga ega bo‘lamiz.

(29) va $(\partial/\partial y)G_1=\left(\partial^2/\partial y^2\right)G_1$, $(\partial/\partial x)G_1=-(\partial/\partial\xi)G_2$ munosabatlarni e’tiborga olib

$$\frac{\partial}{\partial y}\int_0^1\int_0^1 G_1(x,y;\xi,\eta)\,d\xi dx=-2\left[\pi(y-\eta)\right]^{-1/2}-2M_0(y,\eta),$$

$$M_0(y,\eta)=\frac{2}{\sqrt{\pi(y-\eta)}}\sum_{m=1}^{+\infty}\left\{\exp\left[-\frac{m^2}{y-\eta}\right]-\exp\left[-\frac{(2m-1)^2}{4(y-\eta)}\right]\right\}$$

ekanligini topamiz, bu yerda $M_0(y,\eta)$ funksiya $\{(y,\eta):0\le\eta<y<1\}$ da uzluksiz differensiallanuvchi hamda $\eta\to y$ da o‘zi va uning hosilasi nolga intiladi. Agar buni hisobga olsak, u holda $(\partial/\partial y)M(y,\eta)$ funksiya, ya’ni (33) integral tenglamaning yadrosi $1/2$ tartibli maxsuslikka ega.

Endi (33) tenglamaning o‘ng tomoni $g_3'(y)$ funksiyani tadqiq qilamiz. (29) va $(\partial/\partial\xi)G_1=-(\partial/\partial x)G_2$ munosabatlardan foydalanib,

$$\int_0^1 G_{1\xi}(x,y;0,\eta)\,dx=-\int_0^1 G_{1\xi}(x,y;1,\eta)\,dx=\left[\pi(y-\eta)\right]^{-1/2}+M_0(y,\eta)$$

ekanligini topamiz. Endi (31) tenglikning o‘ng tomonidagi birinchi va ikkinchi hadlar yig‘indisini $g_4(y)$ bilan belgilaylik. So‘ngra, oxirgi tenglikni inobatga olib $g_4(y)$ funksiyani

$$g_4(y)=\int_0^y\left[\varphi_1(\eta)+\varphi_2(\eta)\right]\left\{\left[\pi(y-\eta)\right]^{-1/2}+M_0(y,\eta)\right\}d\eta$$

ko‘rinishda yozish mumkin. Bu tenglikni y bo‘yicha differensiallab, so‘ngra bo‘laklab integrallasak,

$$g_4'(y)=\frac{1}{\sqrt{\pi y}}\left[\varphi_1(0)+\varphi_2(0)\right]+$$

$$+\int_0^y\left[\frac{\partial}{\partial\eta}\left\{\left[\varphi_1(\eta)+\varphi_2(\eta)\right]\right\}\frac{1}{\sqrt{\pi(y-\eta)}}+\left[\varphi_1(\eta)+\varphi_2(\eta)\right]\frac{\partial}{\partial y}M_0(y,\eta)\right]d\eta$$

(34)

tenglikka ega bo‘lamiz. (31) tenglikning o‘ng tomonidagi uchinchi qo‘shiluvchini $g_5(y)$ bilan belgilab, so‘ngra (29) va

$(\partial/\partial y)G_1=\left(\partial^2/\partial x^2\right)G_1$, $(\partial/\partial x)G_1=-(\partial/\partial\xi)G_2$

munosabatlarni e’tiborga olsak,

$$\frac{\partial}{\partial y}\int_0^1 G_1(x,y;\xi,0)\,dx=\frac{\partial^2}{\partial x^2}\int_0^1 G_1(x,y;\xi,0)\,dx=\frac{\partial}{\partial\xi}\left[G_2(0,y;\xi,0)-G_2(1,y;\xi,0)\right].$$

kelib chiqadi. Oxirgi tenglikka asosan $g_5'(y)$ni

$$g_5'(y)=\int_0^1\tau(\xi)\frac{\partial}{\partial\xi}\left[G_2(1,y;\xi,0)-G_2(0,y;\xi,0)\right]d\xi\,.$$

ko‘rinishda yozish mumkin. So‘ngra oxirgi tenglikni bo‘laklab integrallab va ushbu

$$G_2(1,y;1,0)-G_2(0,y;1,0)=G_2(0,y;0,0)-G_2(1,y;0,0)=\frac{1}{\sqrt{\pi y}}+M_0(y,0)$$

tenglikni e’tiborga olsak,

$$g_5'(y)=\left[\tau(1)+\tau(0)\right]\left[(\pi y)^{-1/2}+M_0(y,0)\right]+$$

$$\int_0^1 \tau'(\xi)\left[G_2(1,y;\xi,0)-G_2(0,y;\xi,0)\right]d\xi \qquad (35)$$

kelib chiqadi. (34) va (35) tengliklarni $g_3'(y)=g_4'(y)+g_5'(y)+\varphi_3'(y)$ ga olib borib qo‘yib, $\tau(\xi),\varphi_j(y)\in C^1[0,1]$, $j=\overline{1,3}$; $\tau(0)=\varphi_1(0)$, $\tau(1)=\varphi_2(0)$ ekanligini, shuningdek, $M_0(y,\eta)$ va $G_2(x,y;\xi,\eta)$ funksiyalarning xossalarini e’tiborga olib, $g_5'(y)\in C[0,1]$ degan xulosaga kelamiz. $(\partial/\partial y)M(y,\eta)$ va $g_3'(y)$ funksiyalarining xossalariga va ikkinchi tur Volterra integral tenglamalar nazariyasiga ko‘ra, (33) integral tenglama yagona yechimga ega. $\omega_1(y)$ funksiya (33) tenglamadan topilgandan so‘ng 1'-masala yechimi $u(x,y)$ funksiyani (28) formula bilan yozish mumkin bo‘ladi. Qo‘yilgan 1-masala to‘liq tadqiq qilindi.

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