O‘ZGARMAS KOEFFITSIYENTLI INTEGRO-DIFFERENSIAL TENGLAMALAR SISTEMASINI YECHISHNING DALAMBER USULI

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A.O.Mamanazarov
O.A.Abdubannopova

Annotatsiya. *Ushbu maqolada bir jinsli bo‘lmagan chiziqli integro-differensial tenglamalar sistemasi uchun Koshi masalasi yechimi Dalamber usulidan foydalanib aniqlangan.*

Quyidagi

(1) $$\begin{cases} D_{mt}^{\alpha}x(t)=aI_{mt}^{\gamma}x(t)+bI_{mt}^{\gamma}y(t)+f_1(t), \\ D_{mt}^{\alpha}y(t)=cI_{mt}^{\gamma}x(t)+dI_{mt}^{\gamma}y(t)+f_2(t), \end{cases} \quad t>m$$

sistemaning

(2) $$\lim_{t\to m} I_{mt}^{1-\alpha}x(t)=x_0, \lim_{t\to m} I_{mt}^{1-\alpha}y(t)=y_0$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini topish masalasini qaraylik, bu yerda D_{mt}^{α} va I_{mt}^{γ} kasr tartibli Riman-Liuvill ma’nosidagi differensial va integral operatorlar bo‘lib[1], quyidagicha aniqlanadi:

$$D_{mt}^{\alpha}z(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_m^t (t-s)^{-\alpha} z(s)ds,$$

$$I_{mt}^{\gamma}z(t)=\frac{1}{\Gamma(\gamma)}\int_m^t (t-s)^{\gamma-1} z(s)ds,$$

$\Gamma(z)$ – Eylerning gamma-funksiyasi[2], $\alpha,\gamma,a,b,c,d,x_0,y_0$ – berilgan haqiqiy sonlar bo‘lib, $0<\alpha<1$, $\gamma>0$; $f_1(t)$ va $f_2(t)$ berilgan funksiyalar, $x(t)$ va $y(t)$ lar esa noma’lum funksiyalar.

$\{(1),(2)\}$ Koshi masalasining yechimini topish bilan shug‘ullanamiz. Dastlab, (1) ning ikkinchi tenglamasini λ songa ko‘paytirib so‘ngra birinchi tenglamaga qo‘shamiz. Natijada quyidagi tenglikka ega bo‘lamiz:

(3) $$D_{mt}^{\alpha}\left[x(t)+\lambda y(t)\right]=(a+\lambda c)I_{mt}^{\gamma}x(t)+(b+\lambda d)I_{mt}^{\gamma}y(t)+f_1(t)+\lambda f_2(t).$$

(3) ni quyidagi ko‘rinishda yozib olamiz:

(4) $$D_{mt}^{\alpha}\left[x(t)+\lambda y(t)\right]=(a+\lambda c)\left[I_{mt}^{\gamma}x(t)+\frac{b+\lambda d}{a+\lambda c}I_{mt}^{\gamma}y(t)\right]+f_1(t)+\lambda f_2(t).$$

λ sonni shunday tanlaylikki, u ushbu $\dfrac{b+\lambda d}{a+\lambda c}=\lambda$ tenglamaning yechimi bo‘lsin. Faraz qilaylik bu tenglama λ_1 va λ_2 ildizlarga ega bo’lsin. Quyidagi hollarni qaraymiz:

a) $\lambda_1 \neq \lambda_2 \in$ Ў,

b) $\lambda_1 = \lambda_2 = \lambda \in$ Ў,

c) $\lambda_1 \neq \lambda_2 \in$ J .

Dastlab a) holni qaraylik. Bu holda (4) tenglama ushbu

(5) $$D_{mt}^{\alpha}\left[x(t)+\lambda_i y(t)\right]=(a+\lambda_i c)\left[I_{mt}^{\gamma}x(t)+\lambda_i I_{mt}^{\gamma}y(t)\right]+f_1(t)+\lambda_i f_2(t),\quad i=1,2$$

ko‘rinishdagi $x(t)+\lambda_i y(t)$ noma’lumlarga nisbatan chiziqli ikkita tenglama ko’rinishini oladi. Soddalik maqsadida quyidagi belgilashlarni kiritaylik:

(6) $$x(t)+\lambda_i y(t)=z(t),\quad f_1(t)+\lambda_i f_2(t)=f_{3,i}(t),\quad a+\lambda_i c=\tilde{\lambda}_i,\quad i=1,2.$$

U holda (5) tenglama quyidagi ko‘rinishga keladi:

(7) $$D_{mt}^{\alpha}z(t)=\tilde{\lambda}_i I_{mt}^{\gamma}z(t)+f_{3,i}(t),\qquad i=1,2.$$

Ushbu

(8) $$I_{mt}^{\alpha}D_{mt}^{\alpha}\varphi(t)=\varphi(t)-(t-m)^{\alpha-1}\Gamma^{-1}(\alpha)I_{mt}^{1-\alpha}\varphi(t)\big|_{t=m}$$

formulani e’tiborga olgan holda (7) tenglikning har ikki tomoniga I_{mt}^{α} operatorni tatbiq qilamiz. (2) shartlarni va kasr-tartibli integral operator yoyilmasini e’tiborga olsak, $z(t)$ noma’lum funksiyaga nisbatan

(9) $$z(t)-\tilde{\lambda}_i\int_m^t \frac{(t-\eta)^{\alpha+\gamma-1}z(\eta)}{\Gamma(\alpha+\gamma)}=f_{4,i}(t)$$

ko‘rinishdagi 2-tur Volterra integral tenglamasiga ega bo‘lamiz. Bu yerda

(10) $$f_{4,i}(t)=I_{mt}^{\alpha}f_{3,i}(t)+(t-m)^{\alpha-1}\Gamma^{-1}(\alpha)I_{mt}^{1-\alpha}z(t)\big|_{t=m}.$$

U holda (9) tenglama quyidagi ko‘rinishga keladi:

(11) $$z(t)-\tilde{\lambda}_i\int_m^t K(t,\eta)z(\eta)d\eta=f_{4,i}(t).$$

Oxirgi tenglama yechimini kema-ket yaqinlashish usulidan foydalanib topamiz. Nolinchi yaqinlashish sifatida $f_{4,i}(t)$ ni qabul qilamiz:

$$z_0(t)=f_{4,i}(t).$$

Birinchi va ikkinchi yaqinlashishlarni mos holda quyidagi formulalar orqali aniqlaymiz:

$$z_1(t)=f_{4,i}(t)+\tilde{\lambda}_i\int_m^t K(t,\eta)f_{4,i}(\eta)d\eta,$$

$$z_2(t)=f_{4,i}(t)+\tilde{\lambda}_i\int_m^t K(t,\tau)\left[f_{4,i}(\tau)+\tilde{\lambda}_i\int_m^{\tau}K(\tau,\eta)f_{4,i}(\eta)d\eta\right]d\tau=$$

$$= f_{4,i}(t) + \tilde{\lambda}_i \int_m^t K(t,\eta) f_{4,i}(\eta)d\eta + \tilde{\lambda}_i^2 \int_m^t K_2(t,\eta) f_{4,i}(\eta)d\eta.$$

n–yaqinlashishni esa

$$z_n = f_{4,i}(t) + \tilde{\lambda}_i \int_m^t \sum_{j=0}^n \tilde{\lambda}_i^{j-1} K_j(t,\eta) f_{4,i}(\eta)d\eta$$

formuladan foydalanib topamiz, bu yerda $K_j(t,\eta)$ – iteratsiyalangan yadrolar bo‘lib,

$$K_1(t,\eta) = K(t,\eta) = (t-s)^{\alpha+\gamma-1}\Gamma^{-1}(\alpha+\gamma),$$

$$K_j(t,\eta) = \int_\eta^t K(t,\tau)K_{j-1}(\tau,\eta)d\tau. \quad j = 2,3,\ldots .$$

Dastlab, $K_2(t,\eta)$ ni hisoblaylik:

$$K_2(t,\eta) = \int_\eta^t K(t,\tau)K(\tau,\eta)d\tau = \int_\eta^t \frac{(t-\tau)^{\alpha+\gamma-1}(\tau-\eta)^{\alpha+\gamma-1}}{\Gamma^2(\alpha+\gamma)}d\tau. \quad j = 2,3,\ldots .$$

Ushbu almashtirishni bajaramiz: $\tau = (t-\eta)s + \eta$. U holda beta va gamma-funksiya xossalaridan foydalansak [2],

$$K_2(t,\eta) = \int_0^1 \frac{(t-\eta)^{2(\alpha+\gamma)-1}(1-s)^{\alpha+\gamma-1} s^{\alpha+\gamma-1}}{\Gamma^2(\alpha+\gamma)}ds = \frac{(t-\eta)^{2(\alpha+\gamma)-1}}{\Gamma^2(\alpha+\gamma)}\int_0^1 s^{\alpha+\gamma-1}(1-s)^{\alpha+\gamma-1}ds =$$

$$= \frac{(t-\eta)^{2(\alpha+\gamma)-1}}{\Gamma^2(\alpha+\gamma)}B(\alpha+\gamma,\alpha+\gamma) = \frac{(t-\eta)^{2(\alpha+\gamma)-1}\Gamma^2(\alpha+\gamma)}{\Gamma^2(\alpha+\gamma)\Gamma\left[2(\alpha+\gamma)\right]} = \frac{(t-\eta)^{2(\alpha+\gamma)-1}}{\Gamma\left[2(\alpha+\gamma)\right]}.$$

Matematik induksiya usulidan ko‘rsatish mumkinki, $K_j(t,\eta)$ uchun quyidagi tenglik o‘rinli:

$$K_j(t,\eta) = \frac{(t-\eta)^{j(\alpha+\gamma)-1}}{\Gamma\left[j(\alpha+\gamma)\right]}, \quad j = 1,2,3\ldots.$$

Endi $R(t,\eta;\tilde{\lambda}_i)$ rezolventani tuzamiz:

$$R(t,\eta;\tilde{\lambda}_i) = \sum_{j=1}^\infty \tilde{\lambda}_i^{j-1} K_j(t,\eta) = \sum_{j=1}^\infty \frac{\tilde{\lambda}_i^{j-1}(t-\eta)^{j(\alpha+\gamma)-1}}{\Gamma\left[j(\alpha+\gamma)\right]} = \sum_{j=0}^\infty \frac{\tilde{\lambda}_i^{j}(t-\eta)^{j(\alpha+\gamma)+\alpha+\gamma-1}}{\Gamma\left[j(\alpha+\gamma)+\alpha+\gamma\right]}.$$

$E_{\alpha,\beta}(z)$– Mittag-Leffler funksiyasining yoyilmasidan foydalansak [3], rezolventani quyidagicha yozishimiz mumkin:

$$R(t,\eta;\tilde{\lambda}_i) = (t-\eta)^{\alpha+\gamma-1}E_{\alpha+\gamma,\alpha+\gamma}\left[\tilde{\lambda}_i(t-\eta)^{\alpha+\gamma}\right].$$

U holda 2-tur Volterra integral tenglamalari nazariyasiga [4] asosan, (11) tenglama yechimi ushbu formula orqali aniqlanadi:

$$z(t) = f_{4,i}(t) + \tilde{\lambda}_i \int_m^t (t-\eta)^{\alpha+\gamma-1} E_{\alpha+\gamma,\alpha+\gamma}\left[\tilde{\lambda}_i (t-\eta)^{\alpha+\gamma}\right] f_{4,i}(\eta)\,d\eta . \tag{12}$$

Endi (6) va (12) tengliklarga asosan quyidagi tenglamalar sistemasini tuzamiz:

$$x(t) + \lambda_i y(t) = f_{4,i}(t) +$$

$$+\tilde{\lambda}_i \int_m^t (t-\eta)^{\alpha+\gamma-1} E_{\alpha+\gamma,\alpha+\gamma}\left[\tilde{\lambda}_i (t-\eta)^{\alpha+\gamma}\right] f_{4,i}(\eta)\,d\eta , \; i = 1,2 . \tag{13}$$

Oxirgi tenglamar sistemasini algebraik qo'shish usulidan foydalanib, $x(t)$ va $y(t)$ noma'lum funksiyalarni

$$x(t) = f_{4,1}(t) + (a+\lambda_1 c)\int_m^t (t-\eta)^{\alpha+\gamma-1} E_{\alpha+\gamma,\alpha+\gamma}\left[(a+\lambda_1 c)(t-\eta)^{\alpha+\gamma}\right] f_{4,1}(\eta)\,d\eta - \lambda_1 y(t),$$

$$y(t) = \frac{1}{\lambda_1 - \lambda_2}\Big\{ f_{4,1}(t) - f_{4,2}(t) +$$

$$+(a+\lambda_1 c)\int_m^t (t-\eta)^{\gamma-1} E_{\alpha+\gamma,\alpha+\gamma}\left[(a+\lambda_1 c)(t-\eta)^{\alpha+\gamma}\right] f_{4,1}(\eta)\,d\eta -$$

$$-(a+\lambda_2 c)\int_m^t (t-\eta)^{\alpha+\gamma-1} E_{\alpha+\gamma,\alpha+\gamma}\left[(a+\lambda_2 c)(t-\eta)^{\alpha+\gamma}\right] f_{4,2}(\eta)\,d\eta \Big\}$$

ko'rinishda bir qiymatli aniqlashimiz mumkin.

b) holni qaraylik. Bu holda (13) tenglamalar sistemasida quyidagi ikki noma'lumli bitta chiziqli differensial tenglamaga ega bo'lamiz:

$$x(t) + \lambda y(t) = f_4(t) + \tilde{\lambda} \int_m^t (t-\eta)^{\alpha+\gamma-1} E_{\alpha+\gamma,\alpha+\gamma}\left[\tilde{\lambda}(t-\eta)^{\alpha+\gamma}\right] f_4(\eta)\,d\eta .$$

Oxirgidan $y(t)$ noma'lum funksiyani

$$y(t) = \frac{1}{\lambda}\left[f_4(t) + \tilde{\lambda} \int_m^t (t-\eta)^{\alpha+\gamma-1} E_{\alpha+\gamma,\alpha+\gamma}\left[\tilde{\lambda}(t-\eta)^{\alpha+\gamma}\right] f_4(\eta)\,d\eta \right] - \frac{x}{\lambda}$$

ko'rinishda topamiz. So'ngra topilgan $y(t)$ funksiyani (1) tenglamalar sistemaning birinchi tenglamasiga qo'yib

$$D_{mt}^{\alpha} x(t) = \left(a - \frac{b}{\lambda}\right) I_{mt}^{\gamma} x(t) +$$

$$+\frac{b}{\lambda} I_{mt}^{\gamma}\left[f_4(t) + \tilde{\lambda} \int_m^t (t-\eta)^{\alpha+\gamma-1} E_{\alpha+\gamma,\alpha+\gamma}\left[\tilde{\lambda}(t-\eta)^{\alpha+\gamma}\right] f_4(\eta)\,d\eta \right] + f_1(t) \tag{14}$$

tenglamani topamiz. Soddalik maqsadida

$$f_5(t) = \frac{b}{\lambda} I_{mt}^{\gamma}\left[f_4(t) + \tilde{\lambda} \int_m^t (t-\eta)^{\alpha+\gamma-1} E_{\alpha+\gamma,\alpha+\gamma}\left[\tilde{\lambda}(t-\eta)^{\alpha+\gamma}\right] f_4(\eta)\,d\eta \right] + f_1(t),$$

(15) $$\lambda' = a - b\lambda^{-1}.$$

belgilashlarni kiritganimizdan so'ng, (14) tenglama quyidagi ko'rinishni oladi:

$$D_{mt}^{\alpha} x(t) = \lambda' I_{mt}^{\gamma} x(t) + f_5(t).$$

Oxirgi tenglamaning har ikki tarafiga (8) formulani qo'llaymiz. Natijada quyidagi

(16) $$x(t) - \lambda' \int_m^t \frac{(t-\eta)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} x(\eta) d\eta = f_6(t)$$

ko'rinishdagi 2-tur Volterra integral tenglamasiga ega bo'lamiz. Bu yerda

(17) $$f_6(t) = \frac{(t-m)^{\alpha-1}}{\Gamma(\alpha)} I_{mt}^{1-\alpha} x(t)\Big|_{t=m} + I_{mt}^{\alpha} f_5(t).$$

(16) tenglamaning yechimi (9) tenglamaning yechimi kabi topiladi.

Endi c) holni qaraylik. Bu holda {(1),(2)} Koshi masalasining yechimi a) holdagi kabi aniqlanadi.

FOYDALANILGAN ADABIYOTLAR:

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