UMUMLASHGAN HILFER MA’NOSIDAGI HOSILA VA RIMAN-LIUVILL MA’NOSIDAGI INTEGRALNI O‘Z ICHIGA OLUVCHI ODDIY DIFFERENSIAL TENGLAMA UCHUN KOSHI MASALASI

ЗАДАЧА КОШИ ДЛЯ ОБЫКНОВЕННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С ПРОИЗВОДНОЙ В ОБОБЩЕННОМ СМЫСЛЕ ГИЛЬФЕРА И ИНТЕГРАЛОМ В СМЫСЛЕ РИМАНА-ЛИУВИЛЯ

THE CAUCHY PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION INVOLVING THE DERIVATIVE IN THE GENERALIZED HILFER SENSE AND THE INTEGRAL IN THE RIEMAN-LIOUVILLE SENSE

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Annotatsiya. *Ushbu maqolada umumlashgan hilfer ma’nosidagi hosila va riman-liuvill ma’nosidagi integralni o‘z ichiga oluvchi oddiy differensial tenglama uchun koshi masalasi o`rganilgan. Masalaning yechimi Mittag-Leffler funksiyasi yordamida topilgan.Berilgan funksiya yetarli shart topilgan va teorema shaklida bayon qilingan hamda bu teorema isbotlangan.*

Аннотация: *В статье исследуется задача Коши для обыкновенного дифференциального уравнения, содержащего производную в обобщенном смысле Гильфера и интеграл в смысле Римана-Лиувилля. Решение задачи было найдено с помощью функции Миттаг-Леффлера, для данной функции найдено достаточное условие, которое сформулировано в виде теоремы, и эта теорема доказана.*

Abstract: *In this article, the Cauchy problem for an ordinary differential equation containing a derivative in the generalized Hilfer sense and an integral in the Riemann-Liouville sense is studied. The solution to the problem was found using the Mittag-Leffler function. A sufficient condition was found for the given function and it was stated in the form of a theorem, and this theorem was proved.*

Kalit so`zlar: *oddiy differensial tenglama, kasr tartibli operator, Koshi masalasi.*

Ключевые слова: *обыкновенное дифференциальное уравнение, оператор дробного порядка, задача Коши.*

Keywords: *ordinary differential equation, fractional order operator, Cauchy problem .*

I.Kirish. So‘ngi vaqtlarda kasr tartibli integro-differensial operatorlar qatnashgan differensial tengalamalar bilan shug‘ullanishga bo`lgan qiziqish ortib bormoqda. Bunga sabab ko`plab issiqlik tarqalish va diffuziya jarayonlarini matematik modelini tuzish funksiyani biror qiymati qatnashgan differensial tenglama uchun qo‘yiladigan masalalarga keltiriladi. Kasr tartibli integro-differensial operatorlar qatnashgan xususiy hosilali va oddiy differensial tenglamalar ko`plab tadqiqotchilar tomonidan o‘rganilgan.Masalan [1-3] ishlarga qaralsin.

II. Masalaning qo‘yilishi.

(0,1) oraliqda ushbu

$$D_{0x}^{(\alpha,\varphi),\beta}y(x)-\lambda I_{0x}^{\gamma}y(x)=f(x) \qquad (1)$$

kasr tartibli oddiy differensial tenglamani qaraylik, bu yerda $y(x)$-noma’lum funksiya, $f(x)$–berilgan funksiya; $\alpha,\varphi,\beta,\lambda$-o‘zgarmas haqiqiy sonlar bo‘lib, $0<\alpha<1, 0<\varphi<1, 0\le\beta\le 1$; $D_{0x}^{(\alpha,\varphi),\beta}y(x)$-Hilfer ma’nosidagi kasr tartibli hosilasi [4].

$$D_{0x}^{(\alpha,\varphi),\beta}y(x)=I_{0x}^{\beta(1-\alpha)}\frac{d}{dx}I_{0x}^{(1-\beta)(1-\varphi)}y(x)\ , \qquad (2)$$

$I_{0x}^{\gamma}y(x)$–Riman-Liuvill ma’nosida γ (kasr) tartibli integral [5].

$$I_{0x}^{\gamma}y(x)=\frac{1}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}y(t)dt,\ \gamma>0.$$

Koshi masalasi. Shunday $y(x)$ funksiya topilsinki, u quyidagi xossalarga ega bo‘lsin:

1) $x^{(1-\beta)(1-\varphi)}y(x)\in C[0,1]$, $D_{0x}^{(\alpha,\varphi),\beta}y(x)\in C(0,1)$ sinfga tegishli

2) (1) tenlamani qanoatlantirsin;

3) $x=0$ nuqtada esa

$$\lim_{x\to 0} I_{0x}^{(1-\beta)(1-\varphi)}y(x)=A \qquad (3)$$

shartni qanoatlantirsin, bu yerda, A - berilgan o‘zgarmas haqiqiy son.

(1) tenglamani

$$I_{0x}^{\beta(1-\alpha)}\frac{d}{dx}I_{0x}^{(1-\beta)(1-\varphi)}y(x)-\lambda I_{0x}^{\gamma}y(x)=f(x) \qquad (4)$$

ko‘rinishda yozib olamiz.

(4) tenglamaga $D_{0x}^{\beta(1-\alpha)}$ ni ta'sir ettirib,

$$D_{0x}^{\beta(1-\alpha)}I_{0x}^{\gamma}y(x)=\frac{1}{\Gamma(1-\beta(1-\alpha)+\gamma)}\frac{d}{dx}\int_0^x (x-z)^{\gamma-\beta(1-\alpha)}y(z)dz\ ,$$

$$D_{0x}^{\beta(1-\alpha)}y(x)=\frac{1}{\Gamma(1-\beta(1-\alpha))}\frac{d}{dx}\int_0^x (x-z)^{-\beta(1-\alpha)}y(z)dz\ ,$$

formulalardan foydalanib, uni quyidagicha yozib olamiz;

$$\frac{d}{dx}I_{0x}^{(1-\beta)(1-\alpha)}y(x)-\frac{\lambda}{\Gamma(1-\beta(1-\alpha)+\gamma)}\frac{d}{dx}\int_0^x (x-z)^{\gamma-\beta(1-\alpha)}y(z)dz=$$

$$=\frac{1}{\Gamma(1-\beta(1-\alpha))}\frac{d}{dx}\int_0^x (x-z)^{-\beta(1-\alpha)}f(z)dz\ . \qquad (6)$$

(6) ni 0 dan x gacha integrallab, uni quyidagicha yozib olamiz;

$$I_{0x}^{(1-\beta)(1-\alpha)}y(x)-\frac{\lambda}{\Gamma(1-\beta(1-\alpha)+\gamma)}\int_0^x (x-z)^{\gamma-\beta(1-\alpha)}y(z)dz=$$

$$\frac{1}{\Gamma(1-\beta(1-\alpha))}\int_0^x (x-z)^{-\beta(1-\alpha)}f(z)dz+A \qquad (7)$$

(7) ga $D_{0x}^{(1-\beta)(1-\varphi)}$ ni ta’sir ettirib,ba'zi hisoblashlarni amalga oshirib, quyidagi

$$y(x)-\frac{\lambda}{\Gamma(\varphi+\gamma+\beta(\alpha-\varphi)}\int_0^x (x-z)^{\varphi+\gamma+\beta(\alpha-\varphi)-1}y(z)dz=$$

$$=\frac{1}{\Gamma(\varphi+\beta(\alpha-\varphi)}\int_0^x (x-z)^{\varphi+\beta(\alpha-\varphi)-1}f(z)dz+\frac{Ax^{-(1-\beta)(1-\varphi)}}{\Gamma(1-(1-\beta)(1-\varphi))} \qquad (8)$$

ko‘rinishdagi ikkinchi tur Volterra integral tenglamasini hosil qilamiz.

(8) integral tenglamani yechish uchun ba’zi belgilashlarni kiritamiz:

$$g(x)=\frac{1}{\Gamma(\varphi+\beta(\alpha-\varphi))}\int_0^x (x-z)^{\varphi+\beta(\alpha-\varphi)-1}f(z)dz+\frac{Ax^{-(1-\beta)(1-\varphi)}}{\Gamma(1-(1-\beta)(1-\varphi))} \qquad (9)$$

$$K(x,z)=\frac{(x-z)^{\varphi+\gamma+\beta(\alpha-\varphi)-1}}{\Gamma(\varphi+\gamma+\beta(\alpha-\varphi)}. \qquad (10)$$

(9) va (10) tenglamani ketma-ket yaqinlashish usuli orqali yechamiz.

$$K_1(x,z)=\frac{(x-z)^{\varphi+\gamma+\beta(\alpha-\varphi)-1}}{\Gamma(\varphi+\gamma+\beta(\alpha-\varphi)} \quad \text{va} \quad K_i(x,y)=\int_y^x K_1(x,t)K_{i-1}(t,y)dt$$

formulalardan foydalanib, ba’zi hisoblashlarni amalga oshirib, $n-$ iteratsialangan yadroni

$$K_n(x,z)=\frac{(x-z)^{n(\varphi+\gamma+\beta(\alpha-\varphi))-1}}{\Gamma(n(\varphi+\gamma+\beta(\alpha-\varphi)))}$$

ko‘rinishda topamiz. $K_n(x,z)$ orqali $R(x,z,\lambda)$ rezolventani

$$R(x,z,\lambda)=\sum_{n=1}^{+\infty}\frac{\lambda^{n-1}(x-z)^{n(\varphi+\gamma+\beta(\alpha-\varphi))-1}}{\Gamma\left(n\left(\varphi+\gamma+\beta(\alpha-\varphi)\right)\right)}$$

ko‘rinishda topamiz.

Integral tenglamalar nazariyasiga ko‘ra tenglama yechimini,

$$y(x)=g(x)-\lambda\int_0^x R(x,z,\lambda)g(z)dz$$

ko‘rinishda yozib olamiz, bu yerda

$$R(x,z,\lambda)=\sum_{n=1}^{+\infty}\frac{\lambda^{n-1}(x-z)^{n(\varphi+\gamma+\beta(\alpha-\varphi))-1}}{\Gamma\left(n\left(\varphi+\gamma+\beta(\alpha-\varphi)\right)\right)}.$$

(9) va (10) belgilashlarni e’tiborga olib, ba'zi hisoblashlarni amalga oshirib, A masalaning formal yechimini

$$y(x)=Ax^{-(1-\varphi)(1-\beta)}E_{\varphi+\gamma+\beta(\alpha-\varphi),1-(1-\beta(1-\varphi)}[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}]+$$

$$+\int_0^x (x-z)^{\varphi+\beta(\alpha-\varphi)-1} E_{\varphi+\gamma+\beta(\alpha-\varphi),\varphi+\beta(\alpha-\varphi)}[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}] f(z)dz \qquad (11)$$

ko'rinishda topamiz, bu yerda

$$E_{p,q}(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(pn+q)} \qquad (12)$$

- Mittag-Leffer funksiyasi [6].

Teorema. Agar $f(x)=x^{-p} f_1(x)$ bunda $p<1-\mu(1-\alpha)$, $f_1(x)\in C[0,1]$ bo‘lsa, u holda Koshi masalasining yechimi mavjud va u (11) formula bilan aniqlanadi.

Isbot. Dastlab (11) formulani (1) tenglamani qanoatlantirishini ko‘rsatamiz. masala yechimni

$$y(x)=y_1(x)+y_2(x) \qquad (13)$$

ko‘rinishda yozib olamiz, bu yerda

$$y_1(x)=Ax^{-(1-\varphi)(1-\beta)}E_{\varphi+\gamma+\beta(\alpha-\varphi),1-(1-\beta(1-\varphi)}[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}],$$

$$y_2(x)=\int_0^x (x-z)^{\varphi+\beta(\alpha-\varphi)-1} E_{\varphi+\gamma+\beta(\alpha-\varphi),\varphi+\beta(\alpha-\varphi)}[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}] f(z)dz.$$

Avval $y_1(x)$ni (1) tenglamani qanoatlantirishini ko‘rsatamiz.Buning uchun $I_{0x}^{(1-\alpha)(1-\beta)}y_1(x)$ ni hisoblaymiz:

(14)

$$I_{0x}^{(1-\alpha)(1-\beta)}y_1(x)=I_{0x}^{(1-\alpha)(1-\beta)}\left[Ax^{-(1-\varphi)(1-\beta)}E_{\varphi+\gamma+\beta(\alpha-\varphi),1-(1-\beta(1-\varphi)}[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}]\right] \qquad (14)$$

da ba’zi hisoblashlarni amalga oshirib,

$$I_{0x}^{(1-\alpha)(1-\beta)}y_1(x)=AE_{\varphi+\gamma+\beta(\alpha-\varphi),1}\left[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}\right] \qquad (15)$$

tenglikni hosil qilamiz.(15) ni differensiallaymiz:

$$\frac{d}{dx}I_{0x}^{(1-\alpha)(1-\beta)}y_1(x)=A\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)-1}E_{\varphi+\gamma+\beta(\alpha-\varphi),\varphi+\gamma+\beta(\alpha-\varphi)}\left[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}\right] \qquad (16)$$

(16) ni har ikki tarafini $I_{0x}^{\beta(1-\alpha)}$ ni ta’sir ettiramiz:

$$I_{0x}^{\beta(1-\alpha)}\frac{d}{dx}I_{0x}^{(1-\alpha)(1-\beta)}y_1(x)=I_{0x}^{\beta(1-\alpha)}\left[A\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}E_{\varphi+\gamma+\beta(\alpha-\varphi),\varphi+\gamma+\beta(\alpha-\varphi)}\left[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}\right]\right] \qquad (17)$$

(17) da ba’zi hisoblashlarni amalga oshirib,

$$I_{0x}^{\beta(1-\alpha)}\frac{d}{dx}I_{0x}^{(1-\alpha)(1-\beta)}y_1(x)=\lambda Ax^{\varphi+\gamma+\beta-\beta\varphi-1}E_{\varphi+\gamma+\beta(\alpha-\varphi),\varphi+\gamma+\beta-\beta\varphi}[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}] \qquad (18)$$

tenglikni hosil qilamiz.

Yuqoridagi kabi $I_{0x}^{\gamma}y_1(x)$ ni

$$I_{0x}^{\gamma}y_1(x)=Ax^{\varphi+\gamma+\beta-\beta\varphi-1}E_{\varphi+\gamma+\beta(\alpha-\varphi),\varphi+\gamma+\beta-\beta\varphi}[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}] \qquad (19)$$

ko‘rinishda topamiz.

(18) va (19) dan

$$I_{0x}^{\beta(1-\alpha)}\frac{d}{dx}I_{0x}^{(1-\beta)(1-\varphi)}y_1(x)-\lambda I_{0x}^{\gamma}y_1(x)=0 \qquad (20)$$

ekanligi kelib chiqadi.

Endi $y_2(x)$ ni (1) tenglamani qanoatlantirishini ko‘rsatamiz.

Ko‘rsatish qiyin emaski,quyidagi tenglik o‘rinli bo‘ladi:

$$I_{0x}^{(1-\beta)(1-\varphi)}y_2(x)=\int_0^x(x-z)^{-\beta(1-\alpha)}E_{\varphi+\gamma+\beta(\alpha-\varphi),1-(1-\alpha)\beta}[\lambda(x-z)^{\varphi+\gamma+\beta(\alpha-\varphi)}f(z)dz \quad (21)\ (21)$$

da quyidagicha belgilash kiritamiz:

$$I_{0x}^{(1-\beta)(1-\varphi)}y_2(x)=g(x) \quad . \quad (22)$$

Ko‘rish qiyin emaski

$$I_{0x}^{\beta(1-\alpha)}\frac{d}{dx}g(x)=\frac{d}{dx}I_{0x}^{\beta(1-\alpha)+1}g'(x) \quad (23)$$

tenglik o‘rinli bo‘ladi.

(22) va (23) dan foydalanib,

$$\frac{d}{dx}I_{0x}^{\beta(1-\alpha)+1}g'(x)=\frac{d}{dx}\frac{1}{\Gamma(1+\beta(1-\alpha))}\int_0^x(x-t)^{\beta(1-\alpha)}g'(t)dt \quad (24)$$

tenglikni yozib olamiz.

(24) da bir marta bo‘laklab integrallash qoidasini qo‘llab, $g(0)=0$ va $\beta(1-\alpha)>0$ ekanligidan

$$\frac{d}{dx}I_{0x}^{\beta(1-\alpha)+1}g'(x)=\frac{d}{dx}\frac{1}{\Gamma(\beta(1-\alpha))}\int_0^x(x-t)^{\beta(1-\alpha)-1}g(t)dt$$

tenglikni hosil qilamiz.(22) va (21) tengliklardan foydalanib,ba’zi soddalashtirishlarni amalga oshirib,

$$I_{0x}^{\beta(1-\alpha)}\frac{d}{dx}I_{0x}^{(1-\alpha)(1-\beta)}y_2(x)=\frac{d}{dx}\int_0^x f(z)E_{\varphi+\gamma+\beta(\alpha-\varphi),1}[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}]dz \quad (25)$$

tenglikni hosil qilamiz.

Bu yerda $E_{\delta,1}(0)=1$ ekanligini e’tiborga olib,parametr bo‘yicha hosila olib,

$$I_{0x}^{\beta(1-\alpha)}\frac{d}{dx}I_{0x}^{(1-\alpha)(1-\beta)}y_2(x)=f(x)+\lambda\int_0^x(x-t)^{\varphi+\gamma+\beta(\alpha-\varphi)-1}E_{\varphi+\gamma+\beta(\alpha-\varphi),\varphi+\gamma+\beta(\alpha-\varphi)}[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}]f(z)dz$$

(26)

tenglikni hosil qilamiz.

Yuqoridagi kabi $I_{0x}^{\gamma}y_2(x)$ ni

$$I_{0x}^{\gamma}y_2(x)=\int_0^x(x-t)^{\varphi+\gamma+\beta(\alpha-\varphi)-1}E_{\varphi+\gamma+\beta(\alpha-\varphi),\varphi+\gamma+\beta(\alpha-\varphi)}[\lambda x^{\varphi+\gamma+\beta(\alpha-\varphi)}]f(z)dz \quad (27)$$

ko‘rinishda topamiz.

(26) va (27) dan $y_2(x)$ ni (1) tenglamani qanoatlantirishi kelib chiqadi.

(15), (21) va $E_{\delta,1}(0)=1$ dan foydalansak, $\lim_{x\to 0}I_{0x}^{(1-\varphi)(1-\beta)}y(x)=A$ kelib chiqadi.

Teorema isbotlandi.

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