KASR TARTIBLI DIFFUZIYA TENGLAMASI FUNDAMENTAL YECHIMI

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Annotatsiya: Ushbu maqolada kasr tartibli diffuziya tenglamasi fundamental yechimi kasr tartibli differensial tenglama uchun Koshi masalasi yechimi yordamida yechish masalasi hamda bu yechimning klassik yechim bo'lish masalasi o'rganilgan.

Kalit so'zlar: Kasr tartibli diffuziya tenglamasi, Koshi masalasi, Kaputo ma'nosidagi kasr tartibli hosila, Mittag–Leffler funksiyasi, Dirakning delta funksiyasi.

Annotation: In this article, the problem of solving the fundamental solution of the fractional order diffusion equation using the solution of the Cauchy problem for the fractional order differential equation and the problem of this solution being a classical solution are studied.

Keywords: Fractional order diffusion equation, Kaputo fractional derivative, Mittag-Leffler function, Dirac's delta function.

Kirish

Hozirgi zamon matematikasida yangi yo'nalish hisoblangan kasr tartibli hosila, kasr tartibli integral va kasr tartibli differensial tenglamalarni o'rganish dolzarb masala hisoblanadi. Bu mavzuda dunyo olimlari hozirgi kunda juda ko'p izlanishlar olib borishmoqda. Xususan respublikamizda "Romanovskiy" nomidagi matematika instituti Buxoro filialida ustozimiz prof.D.Q.Durdiyev rahbarligida bu mavzu keng o'rganilib kelmoqda.

Ushbu maqolada kasr tartibli differensial tenglamaning kasr tartibli diffuziya tenglamasini yechishdagi tadbiqi nazariy jihatdan o'rganilgan.

Masalaning qo'yilishi: Bizga bir jinsli bo'lmagan kasr tartibli diffuziya tenglamasi berilgan va bu differensial tenglamani kasr tartibli differensial tenglama uchun Koshi masalasi yordamida yechish masalasi qo'yilgan.

$$\begin{cases} {}^{K}D_{0+t}^{\alpha} y(x) + \lambda\, y(x) = f(x) \quad x \epsilon R, t > 0 \\ y(0) = y_o \end{cases} \quad (1)$$ → Koshi masalasi.[3]

$y(x) = y_0 E_\alpha(-\lambda x^{\alpha}) + \int_0^x y^{\alpha-1} E_{\alpha,\alpha}(-\lambda y^\alpha) f(x-y)\,dy$ (2) → Koshi masalasi yechimi.[3]

Kasr tartibli diffuziya tenglamasining fundamental yechimini Koshining (1)masalasi yechimi yordamida keltirib chiqaramiz.

Quyidagi masala qo'yilgan:

$$\begin{cases} {}^{K}D_{0+t}^{\alpha} U(x,t) - U_{xx} = f(x,t) \quad x \epsilon R, t > 0 \\ U(x,0) = \varphi(x) \end{cases} \quad (3)$$

(3)Tenglamaning yechimini topishda Fur'e integral almashtirishlaridan foydalanamiz.[1]

f funksiyaning R dagi Fur'e almashtirishi:

$F[f] = F[f(x)](\text{ʒ}) = \hat{f}(\text{ʒ}) := \int_{-\infty}^{+\infty} e^{ix\text{ʒ}} f(x)dx, \text{ʒ}\epsilon \mathrm{R}$ ko'rinishida bo'ladi.

Kaputo ma'nosidagi kasr tartibli hosila uchun Fur'e integral almashtirishini qo'llaymiz:

$$F\left[{}^{K}D_{0+t}^{\alpha}U\right] = {}^{K}D_{0+t}^{\alpha}\hat{U}$$

U_{xx} ni ham Fur'e integral almashtirishi orqali ifodalaymiz:

$F[U_{xx}]=\int_{-\infty}^{+\infty} e^{ix\text{ʒ}} U_{xx}dx=\int_{\infty}^{-\infty} e^{-ix\text{ʒ}} d(U_x)=$

$=e^{-ix\text{ʒ}}U_x|_{-\infty}^{\infty} + \text{ʒ}i\int_{+\infty}^{-\infty} e^{-ix\text{ʒ}}U_x dx = 0 + \text{ʒ}i\int_{-\infty}^{\infty} e^{-ix\text{ʒ}} dU = \text{ʒ}i(e^{ix\text{ʒ}}U|_{-\infty}^{\infty} +$

$+ \text{ʒ}i\int_{\infty}^{-\infty} e^{ix\text{ʒ}}U d_x = \text{ʒ}i(0 + i\text{ʒ}\hat{U}) = -\text{ʒ}^2\hat{U}$

(Bu yechimga kelishda $\int udv = uv - \int vdu$ bo'laklab integrallash formulasidan, $e^{-ix} = cosx - isinx$, $\lim_{|x|\to\infty}(U_x; U) = 0$ tengliklaridan foydalandik.)

$\mathrm{F}[f(x,t)]\ (\text{ʒ}) = \hat{f}(\text{ʒ}, t)$

Hosil bo'lgan ifodalarni (3) tenglamaga qo'ysak,bu tenglik quyidagi ko'rinishni oladi:

$$\begin{cases} {}^{K}D_{0+t}^{\alpha}\hat{U} + \text{ʒ}^2\hat{U} = \hat{f}(\text{ʒ}, t) & \text{ʒ}\epsilon R, t > 0 \\ \hat{U}(\text{ʒ}, 0) = \hat{\varphi}(\text{ʒ}) \end{cases} \quad (4)$$

Tenglamaning bu ko'rinishi Koshining (1) masalasiga mos keladi:
$$\begin{cases} {}^{K}D_{0+t}^{\alpha}y(x) + \lambda y(x) = f(x) & x\epsilon R, t > 0 \\ y(0) = y_o \end{cases} \quad (1)$$

(3) tenglamaning yechimini topish uchun Koshining (1) masalasi yechimidan foydalanamiz.

$y(x) = y_0 E_\alpha(-\lambda x^\alpha) + \int_0^x y^{\alpha-1}E_{\alpha,\alpha}(-\lambda y^\alpha)f(x-y)dy$ ifoda (1) tenglama yechimi hisoblanadi.

Bunga ko'ra tenglama yechimini keltiramiz:

$$\hat{U}(\text{ʒ}; t) = E_\alpha(-\text{ʒ}^2 t^\alpha) + \int_0^t y^{\alpha-1}E_{\alpha,\alpha}(-\text{ʒ}^2 y^\alpha)\hat{f}(\text{ʒ}; t-y)dy \quad (5)$$

Bu tenglikka ko'ra
$$\begin{cases} {}^{K}D_{0+t}^{\alpha}U(x,t) - U_{xx} = f(x,t) & x\epsilon R, t > 0 \\ U(x,0) = \varphi(x) \end{cases} \quad (3)$$

(3)tenglama yechimiga erishish uchun teskari Furye almashtirishidan foydalanamiz.

$f(x)$ funksiyaning teskari Furye almashtirishi quyidagi integralga aytiladi:

$F^{-1}[f] = F^{-1}[f(x)](\text{ʒ}) = \frac{1}{2\pi}\int_R e^{ix\text{ʒ}}f(\text{ʒ})d\text{ʒ}$

Demak : $$U = F^{-1}[\hat{U}] = \frac{1}{2\pi}\int_R e^{ix\text{ʒ}}\hat{u}(\text{ʒ}; t)d\text{ʒ} \quad (6)$$

(6)tenglikdagi$\hat{U}$ ning o'rniga (5)ifodani qo'ysak (7)ko'rinishidagi kasr tartibli diffuziya tenglamasining fundamental yechimiga ega bo'lamiz:

$$U(x,t) = \frac{1}{2\pi}\int_R e^{ix\text{ʒ}}\left[E_\alpha(-\text{ʒ}^2 t^\alpha) + \int_0^t y^{\alpha-1}E_{\alpha,\alpha}(-\text{ʒ}^2 y^\alpha)\hat{f}(\text{ʒ}; t-y)dy\right]d\text{ʒ} \quad (7)$$

(7)tenglik (3) tenglamaning yechimi hisoblanadi.

Bu yechimning quyidagi (8)masala uchun o'rinli bo'lishini ko'rib chiqamiz:

$$\begin{cases} {}^{K}D_{0+t}^{\alpha}U(x,t) - U_{xx} = f(x,t) \quad x\epsilon R, t > 0 \\ U(x,0) = \delta(x) \end{cases} \qquad (8)$$

Bu yerda δ(x) Dirakning delta funksiyasi:

$$\delta(x - x_o) = \begin{cases} +\infty, & x = x_o \\ 0, & x \neq x_o \end{cases}$$

$\int_R \delta(x - x_o)dx = 1 \quad , \quad \forall y \in D(R) = C_0^{\infty}$(R)

$$\int_R \delta(x - x_o)\varphi(x)dx = \varphi(x_0) \qquad (9)$$

$\delta(x) = \delta(-x)$(juft funksiya)

(7) tenglik bo'yicha $U(x,0)$ni topamiz:

U(x,0)=$\frac{1}{2\pi}\int_R e^{ix\mathfrak{z}}\left[E_\alpha(-\mathfrak{z}^2t^\alpha) + \int_0^0 y^{\alpha-1}E_{\alpha,\alpha}(-\mathfrak{z}^2y^\alpha)\hat{f}(\mathfrak{z}; 0 - y)dy\right]d\mathfrak{z} =$
$\frac{1}{2\pi}\int_R e^{ix\mathfrak{z}}d\mathfrak{z}$ =$\delta(x)$

(bu natija $\int_0^0 y^{\alpha-1}E_{\alpha,\alpha}(-\mathfrak{z}^2y^\alpha)\hat{f}(\mathfrak{z}; 0 - y)dy = 0,$

$E_\alpha(-\mathfrak{z}^2t^\alpha) = \sum_{n=0}^{\infty}\frac{(-\mathfrak{z}^2t^\alpha)^n}{\Gamma(\alpha n+1)}$ ifodalarga ko'ra kelib chiqdi)

(9)formulaga ko'ra (7) tenglik bilan ifodalangan yechimning (8) ko'rinishidagi masala uchun o'rinli ekanligi kelib chiqadi:

U(x,0)=$\int_R \varphi(y)[\frac{1}{2\pi}\int_R e^{i\mathfrak{z}(x-y)}d\mathfrak{z}]dy = \int_R \varphi(y)\ \delta(x-y)dy = \varphi(x)$

Bunga ko'ra

$$\begin{cases} {}^{K}D_{0+t}^{\alpha}U(x,t) - U_{xx} = f(x,t) \quad x\epsilon R, t > 0 \\ U(x,0) = \varphi(x) \end{cases} \qquad (3)$$

tenglamaning

$$U(x,t) = \frac{1}{2\pi}\int_R e^{ix\mathfrak{z}}\left[E_\alpha(-\mathfrak{z}^2t^\alpha) + \int_0^t y^{\alpha-1}E_{\alpha,\alpha}(-\mathfrak{z}^2y^\alpha)\hat{f}(\mathfrak{z}; t - y)dy\right]d\mathfrak{z} \qquad (7)$$

yechimi klassik yechim bo'lishi kelib chiqadi. Demak, tenglamaning bu yechimidan boshqa shu tipdagi fizika-matematika tenglamalari yechishda foydalanish mumkin.

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