SELF-SIMILAR SOLUTIONS OF A CROSS-DIFFUSION SYSTEM OF A NON-DIVERGENT TYPE WITH AN INHOMOGENEOUS DENSITY

D.I Yarmetova
G.O.Ubaydullayeva
D.Sh.Jo'rayeva
National University of Uzbekistan, Tashkent
E-mail: saidovadilafruz1996@mail.ru, ubaydullayevagulnora@mail.ru, dildorajurayeva2021@gmail.com

Abstract: *This work is devoted to numerical solutions of the cross-diffusion system. A non-divergence non-divergence non-linear parabolic type equation system with non-uniform density is considered. The paper presents a self-similar solution of this system, constructs a numerical scheme, and defines an iterative process. Using the implicit scheme, the system is solved by numerical methods.*

Keywords: *cross-diffusion, self-similar solution, numerical solution, front, linearization, iterative Picard method*

Introduction. Let us consider a mathematical model in the domain of a parabolic system of two non-divergent non-linear equations describing the cross-diffusion process, given by the second kind of boundary condition:

$$\begin{cases} \rho(x)\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(v^{m_1-1}\dfrac{\partial u}{\partial x}\right) \\ \rho(x)\dfrac{\partial v}{\partial t} = \dfrac{\partial}{\partial x}\left(u^{m_2-1}\dfrac{\partial v}{\partial x}\right) \end{cases}, \ (x,t) \in R_+ \times (0,+\infty), \qquad (1)$$

With initial conditions (Cauchy problem)

$$\begin{matrix} u(0,x) = u_0(x) \\ v(0,x) = v_0(x) \end{matrix}, \ x \in R_+ \qquad (2)$$

And the boundary conditions:

$$\begin{cases} v^{m_1-1}\dfrac{\partial u}{\partial x}\Big|_{x=0} = u^{q_1}(0,t) \\ u^{m_2-1}\dfrac{\partial v}{\partial x}\Big|_{x=0} = v^{q_2}(0,t) \end{cases}, \quad t > 0 \qquad (3)$$

Here , $m_1, m_2 > 1$, $\rho(x) = |x|^n$ - density, $u(t,x), v(t,x)$ - desired solution

Cross-diffusion means that the spatial movement of one object, described by one of the variables, occurs due to the diffusion of another object, described by another variable. In our problem, the system describes heat conduction in two media with variable density, boundary value problems are set for the flow.

The problem plays an important role in the description of physical, chemical and biological processes such as heat conduction problems, filtration, cross-diffusion transport, dynamics of population systems.

Other tasks were handled by Zhi-wen Duan, Haihua Lu and Li Zhou. And also from our scientists prof. Aripov M, Rakhmonov Z, Sadullaeva Sh, Matyakubov A, Raimbekov Zh works are devoted to the cross-diffusion system.

The peculiarity of this problem is that the system of equations is given with boundary conditions. So far, only the Cauchy problem has been considered.

It is known that systems of degenerate equations may not have a classical solution in the region where $u, \upsilon \equiv 0$. In this case, we study the generalized solution of system (1), which has a physical meaning in the class

$$u(x,t),\, \upsilon(x,t) \ge 0,\, \upsilon^{m_1-1}\frac{\partial u}{\partial x},\, u^{m_2-1}\frac{\partial \upsilon}{\partial x} \in C\left(R_+ \times (0,+\infty)\right)$$

and satisfying system (1) in the sense of distribution [1, 3].

A large number of works [5–15] are devoted to the study of the conditions for global solvability and unsolvability of problem (1)–(3) for various values of numerical parameters (for details, see the bibliography [6]). The authors of [8, 9] studied the conditions for the global solvability and unsolvability in time of the solution and established an estimate for the solution near the explosion time of the nonlocal diffusion problem

$$u_t = u_{xx},\quad \upsilon_t = \upsilon_{xx},\quad x>0,\ 0<T<0, \qquad (4)$$

$$-u_x(0,t) = u^{\alpha}\upsilon^{p},\ \ -\upsilon_x(0,t) = u^{q}\upsilon^{\beta},\ \ 0<t<T, \qquad (5)$$

$$u(x,0) = u_0(x),\, \upsilon(x,0) = \upsilon_0(x),\, x>0. \qquad (6)$$

We proved that if $pq \le (1-\alpha)(1-\beta)$, then any solution to problem (4)-(6) is global.

In [10], the following problems were studied

$$u_t = \left(u^{k_1}\right)_{xx},\quad \upsilon_t = \left(\upsilon^{k_2}\right)_{xx},\quad x\in R_+,\ t>0, \qquad (7)$$

$$-\left(u^{k_1}\right)_x(0,t) = \upsilon^{p}(0,t),\quad -\left(\upsilon^{k_2}\right)_x(0,t) = u^{q}(0,t),\quad t>0, \qquad (8)$$

$$u(x,0) = u_0(x),\quad \upsilon(x,0) = \upsilon_0(x),\quad x\in R_+, \qquad (9)$$

It is shown that the solution of problem (7)-(8) is global if $pq \le (k_1+1)(k_2+1)/4$. Conditions were obtained for the numerical parameters of systems (7)-(9), under which the solution of the problem explodes in a finite time.

Also noteworthy is the work [11], where system (7) was studied with the following boundary conditions

$$-\left(u^{k_1}\right)_x(0,t) = u^{\alpha}\upsilon^{p}(0,t),\quad -\left(\upsilon^{k_2}\right)_x(0,t) = u^{q}\upsilon^{\beta}(0,t),\quad t>0.$$

Proved that $\min\{y_1-r_1, y_2-r_2\} = 0$ where

$$r_1 = \frac{2p + k_2 + 1 - 2\beta}{4pq - (k_2 + 1 - 2\alpha)(k_1 + 1 - 2\beta)},$$

$$r_2 = \frac{2p + k_1 + 1 - 2\beta}{4pq - (k_2 + 1 - 2\alpha)(k_1 + 1 - 2\beta)},$$

$y_1 = \frac{1 - r_1(k_1 - 1)}{2}$, $y_2 = \frac{1 - r_2(k_2 - 1)}{2}$ is a Fujita-type critical exponent.

This work is devoted to the study of the asymptotics of the self-similar solution of problem (1)-(3). Various self-similar solutions of problem (1)-(3) are constructed for the case of slow diffusion $m_1, m_2 > 1$, which are the asymptotics of the solutions of the problem under consideration. For a numerical study, methods are proposed for choosing an appropriate initial approximation for an iterative process that preserves the qualitative properties of problem (1)-(3). An iterative process was also constructed and numerical calculations were carried out, showing fast convergence to the exact solution.

$$\begin{cases} \underline{u}(x,t) = (T+t)^{-\alpha_1} f(\xi), \\ \underline{\upsilon}(x,t) = (T+t)^{-\alpha_2} \varphi(\xi), \; \xi = x(T+t)^{-\gamma} \end{cases}$$

The system of equations (1) at describes processes $m_i > 1 \; (i = 1, 2)$ with a finite perturbation propagation velocity. Equations (1) with $u(x,t)$, $\upsilon(x,t) = 0$, are degenerate, so problem (1)-(3) admits a generalized solution that does not have the necessary smoothness at the points of degeneration.

The self-similar solution to this equation is

$$u = (T+t)^{-\alpha_1} f(\xi_1) \qquad v = (T+t)^{-\alpha_2} f(\xi_2)$$

$$\xi_1 = \frac{x}{(T+t)^{\gamma_1}} \qquad \xi_2 = \frac{x}{(T+t)^{\gamma_2}} \qquad (10)$$

$$\alpha_1, \alpha_2, \gamma_1, \gamma_2$$

Where $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are the unknown coefficients. We put (4) into the system of equations (1) and obtain the following system of equations:

$$\begin{cases} \xi_1^{\,n} (T+t)^{\gamma_1 n} (T+t)^{-\alpha_1 - 1} (-\alpha_1 f(\xi_1) - \gamma_1 \xi_1 \frac{\partial f}{\partial \xi_1}) = \frac{\partial}{\partial x}\left((T+t)^{-\lambda_2 (m_1 - 1)} f^{m_1 - 1}(\xi_2) (T+t)^{-\alpha_1 - \gamma_1} \frac{\partial f}{\partial \xi_1} \right) \\ \xi_2^{\,n} (T+t)^{\gamma_2 n} (T+t)^{-\alpha_2 - 1} (-\alpha_2 f(\xi_2) - \gamma_2 \xi_2 \frac{\partial f}{\partial \xi_2}) = \frac{\partial}{\partial x}\left((T+t)^{-\lambda_1 (m_2 - 1)} f^{m_2 - 1}(\xi_1) (T+t)^{-\alpha_2 - \gamma_2} \frac{\partial f}{\partial \xi_2} \right) \end{cases}$$

(11)

We solve the following two systems of equations with four unknowns

$$\begin{cases} \gamma_1 n - \alpha_1 - 1 = -\alpha_2 (m_1 - 1) - \alpha_1 - 2\gamma_1 \\ \gamma_2 n - \alpha_2 - 1 = -\alpha_1 (m_2 - 1) - \alpha_2 - 2\gamma_2 \end{cases}$$

$$\begin{cases}\gamma_1 n-1=-\alpha_2(m_1-1)-2\gamma_1\\ \gamma_2 n-1=-\alpha_2(m_2-1)-2\gamma_2\end{cases}$$

And find the unknown coefficients

$$\alpha_1=\frac{(1-q_2)(2-n)-(m_1-1)(3-n)}{(m_1-1)(m_2-1)(3-n)^2-(1-q_1)(1-q_2)(2-n)^2}$$
$$\alpha_2=\frac{(1-q_1)(2-n)-(m_2-1)(3-n)}{(m_1-1)(m_2-1)(3-n)^2-(1-q_1)(1-q_2)(2-n)^2}$$
$$\gamma_1=\frac{1}{2-n}-\frac{(m_1-1)}{(2-n)}\frac{(1-q_1)(2-n)-(m_2-1)(3-n)}{(m_1-1)(m_2-1)(3-n)^2-(1-q_1)(1-q_2)(2-n)^2} \quad (12)$$
$$\gamma_2=\frac{1}{2-n}-\frac{(m_2-1)}{(2-n)}\frac{(1-q_2)(2-n)-(m_1-1)(3-n)}{(m_1-1)(m_2-1)(3-n)^2-(1-q_1)(1-q_2)(2-n)^2}$$

From there we get a self-similar system of equations:

$$\begin{cases}\frac{\partial}{\partial\xi_1}\left(f^{m_1-1}(\xi_2)\frac{\partial f}{\partial\xi_1}\right)+\xi_1^{\,n}(\alpha_1 f(\xi_1)+\gamma_1\xi_1\frac{\partial f}{\partial\xi_1})=0\\ \frac{\partial}{\partial\xi_2}\left(f^{m_2-1}(\xi_1)\frac{\partial f}{\partial\xi_2}\right)+\xi_2^{\,n}(\alpha_2 f(\xi_2)+\gamma_2\xi_2\frac{\partial f}{\partial\xi_2})=0\end{cases} \quad (13)$$

We are looking for an unknown function in the form $f=(a-b\xi^{k_1})^{k_2}$. Putting it to system (13) and taking into account the boundary conditions, we obtain:

$a=1$

$b=\frac{\gamma}{n+2}(m_2-1)$

$k_1=n+2$

$k_2=\frac{1}{m_2}-1$

We must not forget that $u(t,x),v(t,x)$ the solution is a finite and $u(t,x)=0$ where $x\geq l(t)$

Here is $l(t)=(T+t)^{\gamma_1}\left(\frac{a}{b}\right)^{\frac{1}{k_1}}$ the front.

NUMERICAL SOLUTION

The area under consideration is divided into parts

$$\Omega=\left\{(x_i,t_j):x_i=ih;h=\frac{l(t)}{n};t_j=j\tau;\tau=\frac{T}{m};i=\overline{0,n};\ j=\overline{0,m}\right\}$$

We replace the system of equations with finite difference schemes

$$\begin{cases}x_i^{\,n}\frac{u_i^j-u_i^{j-1}}{\tau}=\frac{1}{h}\left(a1_{i+1}^j(v)\frac{u_{i+1}^j-u_i^j}{h}-a1_i^j(v)\frac{u_i^j-u_{i-1}^j}{h}\right)\\ x_i^{\,n}\frac{v_i^j-v_i^{j-1}}{\tau}=\frac{1}{h}\left(a2_{i+1}^j(u)\frac{v_{i+1}^j-v_i^j}{h}-a2_i^j(u)\frac{v_i^j-v_{i-1}^j}{h}\right)\end{cases}$$

Where $a_i^j(v)=(v_i^j)^{m_1-1}$

Using the iterative Picard method, we obtain

$$\begin{cases} x_i^n \dfrac{\overset{s}{u}{}_i^j - u_i^{j-1}}{\tau} = \dfrac{1}{h}\left(\left(\overset{s-1}{v}{}_{i+1}^j\right)^{m_1-1} \dfrac{\overset{s}{u}{}_{i+1}^j - \overset{s}{u}{}_i^j}{h} - \left(\overset{s-1}{v}{}_i^j\right)^{m_1-1} \dfrac{\overset{s}{u}{}_i^j - \overset{s}{u}{}_{i-1}^j}{h} \right) \\ x_i^n \dfrac{\overset{s}{v}{}_i^j - v_i^{j-1}}{\tau} = \dfrac{1}{h}\left(\left(\overset{s-1}{u}{}_{i+1}^j\right)^{m_2-1} \dfrac{\overset{s}{v}{}_{i+1}^j - \overset{s}{v}{}_i^j}{h} - \left(\overset{s-1}{u}{}_i^j\right)^{m_2-1} \dfrac{\overset{s}{v}{}_i^j - \overset{s}{v}{}_{i-1}^j}{h} \right) \end{cases}$$

$s = 0,1,2\ldots$ iteration and continues until the condition is met, here $\varepsilon > 0$

We solve the implicit scheme using the sweep method

$$\begin{cases} A_i^1 \overline{u}_{i-1} - C_i^1 \overline{u}_i + B_i^1 \overline{u}_{i+1} = -F_i^1 \\ A_i^1 v_{i-1} - C_i^1 \overline{v}_i + B_i^1 \overline{v}_{i+1} = -F_i^1 \end{cases}$$

Where

$$\begin{cases} A_i^1 = \dfrac{\tau}{h^2}\left(v_i^j\right)^{m_1-1} \\ C_i^1 = \dfrac{\tau}{h^2}\left(\left(v_{i+1}^j\right)^{m_1-1} + \left(v_i^j\right)^{m_1-1}\right) + x_i^n \\ B_i^1 = \dfrac{\tau}{h^2}\left(v_{i+1}^j\right)^{m_1-1} \end{cases} \quad \text{And} \quad \begin{cases} A_i^2 = \dfrac{\tau}{h^2}\left(u_i^j\right)^{m_2-1} \\ C_i^2 = \dfrac{\tau}{h^2}\left(\left(u_{i+1}^j\right)^{m_2-1} + \left(u_i^j\right)^{m_2-1}\right) + x_i^n \\ B_i^2 = \dfrac{\tau}{h^2}\left(u_{i+1}^j\right)^{m_2-1} \end{cases}$$

Conclusion

A self-similar system of equations is constructed for a system of a degenerate parabolic equation of a non-divergence form with a non-uniform density (1) and a self-similar solution of this system is found, depending on the value of the numerical parameters. It is proved that the found solution is the asymptotics of all solutions vanishing at infinity.

The peculiarity of this problem is that the system of equations is given with boundary conditions. So far, only the Cauchy problem has been considered.

For a numerical solution of the considered problem, as initial approximations, one can use the asymptotics of solutions

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