PUASSON TENGLAMASI UCHUN DIRIXLE MASALASINI UMUMLASHTIRISH

Olimova Feruza

Farg`ona davlat universiteti Matematika (yo`nalishlar bo`yicha) yo`nalishi 2-kurs magistranti

Annotatsiya. *Ushbu maqolada biz to`g`ri to`rtburchak sohada Puasson tenglamasi uchun Dirixle masalasini umumlashtirishni o`rganamiz. Noma`lum funksiyaning k-tartibli normal hosilalari to`g`ri to`rtburchakning pastki va yuqori asoslarida, yon tomonlarida esa birinchi turdagi bir jinsli bo`lmagan chegaraviy shartlari berilgan. Bunday holda biz ushbu masala yechimining yagonaligi va mavjudligini isbotlaymiz.*

Kalit so`zlar: *Puasson tenglamasi, Dirixle masalasi, masala yechimining yagonaligi, mavjudligi.*

1.Kirish. Masalani shakllantirish.

Elliptik tenglamalar uchun chegaraviy masalalar ko`plab tadqiqotchilar tomonidan keng o`rganilgan (masalan [1,2]). [3] ishda $(0<x<\infty, t>0)$ sohada issiqlik tarqalish tenglamasi uchun quyidagi

$$\sum_{k=1}^{m} a_k \frac{\partial^k u(0,t)}{\partial x^k} = f(x,t), \qquad u(x,0)=0$$

masala yechimining yagonaligi va mavjudligi isbotlangan. [4] ishda Laplas tenglamasi uchun n o`lchovli chegaralangan D sohada

$$\frac{d^m u}{dv^m} = f(x), \quad x \in \partial D$$

masala o`rganilgan va uni Fredgolmga tegishli ekanligi isbotlangan. Laplas, Puasson va Gelmgolts tenglamalari uchun birlik sharda chegaraviy shartlarda yuqori tartibli hosilalar berilgan masalalar Karachik [5-8], Sokolovskiy [9] va boshqalar tomonidan o`rganilgan. [4-8] ishlarda chegaraviy shartlar butun chegara bo`yicha berilgan. Shuning uchun masala yechimining yagonaligi ma`lum darajadagi bir jinsli ko`phadlar doirasida isbotlangan. Issiqlik tarqalish tenglamasi uchun to`g`ri to`rtburchak sohada boshlang`ich shartida yuqori tartibli hosila berilgan boshlang`ich-chegaraviy masala [10] ishda o`rganilgan. To`g`ri to`rtburchak sohada chegaraviy masalalarni M.S.Azizov o`rgangan ([11-17]).

$\Omega = \{(x,y): 0<x<p, 0<y<q\}$ sohada

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \tag{1}$$

tenglamani qaraymiz.

1-masala. (1) tenglamaning $\frac{\partial^k}{\partial y^k} u(x,y) \in C(\overline{\Omega})$ sinfga tegishli va quyidagi

$$u(0,y)=\psi_1(y),\ 0\le y\le q, \qquad (2)$$

$$u(p,y)=\psi_2(y),\ 0\le y\le q\,, \qquad (3)$$

$$\frac{\partial^k u}{\partial y^k}(x,0)=\bar{\varphi}_{1k},\ 0\le x\le p, \qquad (4)$$

$$\frac{\partial^k u}{\partial y^k}(x,q)=\bar{\varphi}_{2k}(x),\ 0\le x\le p, \qquad (5)$$

shartlarni qanoatlantiruvchi $u(x,y)\in C^{2,2}_{x,y}(\overline{\Omega})$ yechimi topilsin, bu yerda k-fiksirlangan (qat`iy) nomanfiy butun son, $\bar{\varphi}_{1k}(x)$, $\bar{\varphi}_{2k}(x)$, $\psi_1(y)$va $\psi_2(y)$ berilgan funksiyalar.

[18] ishda (2) va (3) chegaraviy shartlar bir jinsli bo`lgan holda o`rganilgan.

1. 1-masala yechimining yagonaligi

1-teorema. Agar 1-masalaning yechimi mavjud bo`lsa u yagonadir.

Isbot. Faraz qilaylik $\varphi_{jk}(x)=0,\ \psi_j(y)=0,\ \ j=\overline{1,2}\ \ 0\le x\le p,\ \ f(x,y)=0,$ $(x,y)\in\overline{\Omega}$ bo`lsin. $\overline{\Omega}$ da $u(x,y)=0$ ekanligini ko`rsatamiz.

$$\alpha_n(y)=\int_0^p u(x,y)X_n(x)dx\,. \qquad (6)$$

integralni kiritamiz. (6) ni y bo`yicha ikki marta differensiallab

$$\alpha_n''(y)=\int_0^p \frac{\partial^2 u}{\partial y^2}X_n(x)dx \qquad (7)$$

ni topamiz. (7) ning o`ng tomonini (3) va (4) shartlarni hisobga olgan holda $y$ bo`yicha ikki marta bo`laklab integrallab

$$a_n''(x)-\lambda^2 a_n(x)=0$$

tenglamani hosil qilamiz.

Uning yechimi

$$a_n(x)=C_{1n}\cos\lambda_n x+C_{2n}\sin\lambda_n x$$

ko`rinishda yoziladi. $C_{1n,}C_{2n}$ noma`lum koeffitsiyentlarni topishda quyidagi

$$a_n^{(k)}(0)=0,\quad a_n^{(k+1)}(0)=0 \qquad (8)$$

ga o`tadigan (4), (5) shartlardan foydalanamiz.

$a_n^{(k)}(x),\ a_n^{(k+1)}(x)$ hosilalar

$$a_n^{(k)}(x)=\lambda_n^k\left[C_{1n}\cos\left(\frac{k\pi}{2}+\lambda_n y\right)+C_{2n}\sin\left(\frac{k\pi}{2}+\lambda_n y\right)\right],$$

$$a_n^{(k+1)}(x)=\lambda_n^{k+1}\left[C_{1n}\cos\left(\frac{(k+1)\pi}{2}+\lambda_n y\right)+C_{2n}\sin\left(\frac{(k+1)\pi}{2}+\lambda_n y\right)\right]$$

ko`rinishga ega.

(8) dan foydalanib $C_{1n,}C_{2n}$ noma`lum koeffitsiyentlarni topishda aniqlovchisi birga teng bo`lgan tenglamalar sistemasini hosil qilamiz:

$$\begin{cases} C_{1n}\cos\frac{k\pi}{2}+C_{2n}\sin\frac{k\pi}{2}=0 \\ C_{1n}\cos\frac{(k+1)\pi}{2}+C_{2n}\sin\frac{(k+1)\pi}{2}=0. \end{cases}$$

Bu yerdan $a_n(x)=0$ ekanligi kelib chiqadi. $X_n(x)$ funksiyaning to`laligiga asosan $\overline{\Omega}$ da $u(x,y)=0$ ekanligi kelib chiqadi.

1-teorema isbotlandi.

2. 1-masala yechimining mavjudligi

(2)-(3) chegaraviy shartlar bir jinsli bo'lmaganligi sababli bu masalani yechish uchun, to'g'ridan to'g'ri o'zgaruvchilari ajralatish usulini qo'llab bo'lmaydi. Lekin xususiy hosilali differensial tenglamalar nazariyasidan ma'lumki, bu masalani bir jinsli chegaraviy masalaga olib kelishimiz mumkin. Shuning uchun quyidagi yordamchi funksiyani kiritamiz:

$$w(x,t)=\frac{x}{p}\left[\psi_2(y)-\psi_1(y)\right]+\psi_1(y) \quad (9)$$

Tekshirib ko'rish qiyin emaski

$$w(0,y)=\psi_1(y); \qquad w(p,y)=\psi_2(y),$$

Masalani yechimini yig'indi ko'rinishda yozamiz

$$u(x,y)=v(x,y)+w(x,y). \quad (10)$$

Bu yerda $v(x,y)$ - yangi noma'lum funksiya. Shunday qilib (10) ga asosan biz keyingi masalaga keldik:

$\Omega=\{(x,y):0<x<p,0<y<q\}$, sohada

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=f_1(x,y) \quad (11)$$

tenglamani qaraylik.

2-masala. (11) tenglamaning $\frac{\partial^k}{\partial y^k}v(x,y)\in C(\overline{\Omega})$ sinfga tegishli va quyidagi

$$v(0,y)=0,\ 0\le y\le q, \quad (12)$$

$$v(p,y)=0,\ 0\le y\le q, \quad (13)$$

$$\frac{\partial^k v}{\partial y^k}(x,0)=\varphi_{1k}(x),\ 0\le x\le p, \quad (14)$$

$$\frac{\partial^k v}{\partial y^k}(x,q)=\varphi_{2k}(x),\ 0\le x\le p, \quad (15)$$

shartlarni qanoatlantiruvchi yechimi topilsin, bu yerda

$$\frac{\partial^k}{\partial y^k}v(x,0)=\frac{\partial^k}{\partial y^k}u(x,0)-\frac{\partial^k}{\partial y^k}w(x,0)\equiv\varphi_{1k}(x),\qquad 0\le x\le p,$$

$$\frac{\partial^k}{\partial y^k}v(x,0)=\frac{\partial^k}{\partial y^k}u(x,0)-\frac{\partial^k}{\partial y^k}w(x,0)\equiv\varphi_{2k}(x),\qquad 0\le x\le p.$$

$$f_1(x,y)=f(x,y)-w_{xx}(x,y)-w_{yy}(x,y).$$

2-masala yechimini

$$u(x,y)=\sum_{n=1}^{\infty}u_n(y)X_n(x). \qquad (16)$$

ko`rinishda izlaymiz.

$f_1(x,y)$, $\varphi_{1k}(x)$, $\varphi_{2k}(x)$ funksiyalarni $X_n(x)$ funksiyalari bo`yicha Furye qatoriga yoyamiz:

$$f_1(x,y)=\sum_{n=1}^{\infty}f_n(y)X_n(x), \qquad (17)$$

$$\varphi_{1k}(x)=\sum_{n=1}^{\infty}\varphi_{1n}X_n(x), \qquad (18)$$

$$\varphi_{2k}(x)=\sum_{n=1}^{\infty}\varphi_{2n}X_n(x), \qquad (19)$$

bu yerda

$$f_n(y)=\int_0^p f_1(x,y)X_n(x)dx, \qquad (20)$$

$$\varphi_{1n}=\int_0^p\varphi(x)X_n(x)dx, \qquad (21)$$

$$\varphi_{2n}=\int_0^p\psi(x)X_n(x)dx. \qquad (22)$$

Ko`rinib turibdiki $v(x,y)$ning yechimi (12), (13) shartlarni qanoatlantiradi. Furye usulini qo`llab 2 - masala yechimini

$$v(x,y)=\sum_{n=1}^{\infty}X_n(x)\left\{\frac{e^{\lambda_n(q-y)}-(-1)^k e^{-\lambda_n(q-y)}}{(-1)^k\,2sh\lambda_n q}\left[\frac{\varphi_n}{\lambda_n^k}-\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\frac{f_n^{(k-2-2s)}(0)}{\lambda_n^{k-2s}}+\right.\right.$$

$$\left.+\frac{1}{2\lambda_n}\int_0^q e^{-\lambda_n\eta}f_n(\eta)d\eta\right]+\frac{(-1)^k e^{\lambda_n y}-e^{-\lambda_n y}}{(-1)^k\,2sh\lambda_n q}\left[\frac{\psi_n}{\lambda_n^k}-\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\frac{f_n^{(k-2-2s)}(q)}{\lambda_n^{k-2s}}+\right.$$

$$\left.+\frac{(-1)^k}{2\lambda_n}\int_0^q e^{-\lambda_n(q-\eta)}f_n(\eta)d\eta\right]-\frac{1}{2\lambda_n}\int_0^y e^{-\lambda_n(y-\eta)}f_n(\eta)d\eta-$$

$$-\frac{1}{2\lambda_n}\int_y^q e^{-\lambda_n(\eta-y)}f_n(\eta)d\eta\}. \qquad (23)$$

ko`rinishda topamiz. Tenglama va chegaraviy shartlarda ishtirok etuvchi hosilalar

$$\frac{\partial^2 v}{\partial x^2}=-\sum_{n=1}^{\infty}X_n(x)\left\{\frac{e^{\lambda_n(q-y)}-(-1)^k e^{-\lambda_n(q-y)}}{(-1)^k 2sh\lambda_n q}\left[\frac{\varphi_n}{\lambda_n^{k-2}}-\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\frac{f_n^{(k-2-2s)}(0)}{\lambda_n^{k-2-2s}}+\right.\right.$$

$$\left.+\frac{1}{2}\sum_{s=0}^{1}\frac{1}{\lambda_n^s}\left(f_n^{(s)}(0)-f_n^{(s)}(q)e^{-\lambda_n q}\right)+\frac{1}{2\lambda_n}\int_0^q f_n''(\eta)e^{-\lambda_n\eta}d\eta\right]+$$

$$+\frac{(-1)^k e^{\lambda_n y}-e^{-\lambda_n y}}{(-1)^k 2sh\lambda_n q}\left[\frac{\psi_n}{\lambda_n^{k-2}}-\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\frac{f_n^{(k-2-2s)}(q)}{\lambda_n^{k-2-2s}}+\right.$$

$$\left.+\frac{(-1)^k}{2}\sum_{s=0}^{1}\frac{(-1)^s}{\lambda_n^s}\left[f_n^{(s)}(q)-f_n^{(s)}(0)e^{-\lambda_n q}\right]+\frac{(-1)^k}{2\lambda_n}\int_0^q f_n''(\eta)e^{-\lambda_n(q-\eta)}d\eta\right]-$$

$$-\frac{1}{2}\sum_{s=0}^{1}\frac{(-1)^s}{\lambda_n^s}\left[f_n^{(s)}(y)-f_n^{(s)}(0)e^{-\lambda_n y}\right]-\frac{1}{2\lambda_n}\int_0^y f_n''(\eta)e^{-\lambda_n(y-\eta)}d\eta-$$

$$\left.-\frac{1}{2}\sum_{s=0}^{1}\frac{1}{\lambda_n^s}\left[f_n^{(s)}(y)-f_n^{(s)}(q)e^{-\lambda_n(q-y)}\right]-\frac{1}{2\lambda_n}\int_y^q f_n''(\eta)e^{-\lambda_n(\eta-y)}d\eta\right\}, \qquad (24)$$

$$\frac{\partial^2 v}{\partial y^2}=\sum_{n=1}^{\infty}X_n(x)\left\{\frac{e^{\lambda_n(q-y)}-(-1)^k e^{-\lambda_n(q-y)}}{(-1)^k 2sh\lambda_n q}\left[\frac{\varphi_n}{\lambda_n^{k-2}}-\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\frac{f_n^{(k-2-2s)}(0)}{\lambda_n^{k-2-2s}}+\right.\right.$$

$$\left.+\frac{1}{2}\sum_{s=0}^{1}\frac{1}{\lambda_n^s}\left(f_n^{(s)}(0)-f_n^{(s)}(q)e^{-\lambda_n q}\right)+\frac{1}{2\lambda_n}\int_0^q f_n''(\eta)e^{-\lambda_n\eta}d\eta\right]+$$

$$+\frac{(-1)^k e^{\lambda_n y}-e^{-\lambda_n y}}{(-1)^k 2sh\lambda_n q}\left[\frac{\psi_n}{\lambda_n^{k-2}}-\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\frac{f_n^{(k-2-2s)}(q)}{\lambda_n^{k-2-2s}}+\frac{(-1)^k}{2}\sum_{s=0}^{1}\frac{(-1)^s}{\lambda_n^s}\left(f_n^{(s)}(q)-\right.\right.$$

$$\left.\left.-f_n^{(s)}(0)e^{-\lambda_n q}\right)+\frac{(-1)^k}{2\lambda_n}\int_0^q f_n''(\eta)e^{-\lambda_n(q-\eta)}d\eta\right]+f_n(y)-\frac{1}{2}\sum_{s=0}^{1}\frac{(-1)^s}{\lambda_n^s}\left[f_n^{(s)}(y)-\right.$$

$$\left.-f_n^{(s)}(0)e^{-\lambda_n y}\right]-\frac{1}{2\lambda_n}\int_0^y f_n''(\eta)e^{-\lambda_n(y-\eta)}d\eta-\frac{1}{2}\sum_{s=0}^{1}\frac{1}{\lambda_n^s}\left[f_n^{(s)}(y)-\right.$$

$$-f_n^{(s)}(q)e^{-\lambda_n(q-y)}\Big]-\frac{1}{2\lambda_n}\int_y^q f_n''(\eta)e^{-\lambda_n(\eta-y)}d\eta\Big\}. \qquad (25)$$

$$\frac{\partial^k v}{\partial y^k}=\sum_{n=1}^{\infty}X_n(x)\Bigg\{\frac{sh\lambda_n(q-y)}{sh\lambda_n q}\Bigg[\varphi_n-\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\lambda_n^{2s}f_n^{(k-2-2s)}(0)+\frac{1}{2}\sum_{s=0}^{k-1}\lambda_n^{k-2-s}\Big[f_n^{(s)}(0)-$$

$$-f_n^{(s)}(q)e^{-\lambda_n q}\Big]+\frac{1}{2\lambda_n}\int_0^q f_n^{(k)}(\eta)e^{-\lambda_n\eta}d\eta\Bigg]+\frac{sh\lambda_n y}{sh\lambda_n q}\Bigg[\psi_n-\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\lambda_n^{2s}f_n^{(k-2-2s)}(q)+$$

$$+\frac{1}{2}\sum_{s=0}^{k-1}\frac{(-1)^{k+s}}{\lambda_n^{s+2-k}}\left(f_n^{(s)}(q)-f_n^{(s)}(0)e^{-\lambda_n q}\right)+\frac{1}{2\lambda_n}\int_0^q f_n^{(k)}(\eta)e^{-\lambda_n(q-\eta)}d\eta\Bigg]+$$

$$+\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\lambda_n^{2s}f_n^{(k-2-2s)}(y)-\frac{(-1)^k}{2}\sum_{s=0}^{k-1}(-1)^s\lambda_n^{k-2-2s}\Big[f_n^{(s)}(y)-f_n^{(s)}(0)e^{-\lambda_n y}\Big]-$$

$$-\frac{1}{2\lambda_n}\int_0^y f_n^{(k)}(\eta)e^{-\lambda_n(y-\eta)}d\eta-\frac{1}{2}\sum_{s=0}^{k-1}\lambda_n^{k-2-s}\Big[f_n^{(s)}(y)-f_n^{(s)}(q)e^{-\lambda_n(q-y)}\Big]-$$

$$-\frac{1}{2\lambda_n}\int_y^q f_n^{(k)}(\eta)e^{-\lambda_n(\eta-y)}d\eta\Big\}. \qquad (26)$$

ko`rinishga ega.

Quyidagi lemmalar o`rinlidir.

(17) - (19) qatorlarning absolyut va tekis yaqinlashuvchi ekanligini isbotlaymiz.

1-lemma. Faraz qilaylik $k\geq 1$ bo`lganda $\varphi_{1k}(x)\in W_2^1(\Omega)$, $\varphi_{1k}(0)=\varphi_{1k}(p)=0$ bo`lsin, u holda

$$\sum_{n=1}^{\infty}X_n(x)\varphi_{1n} \qquad (27)$$

qator $\overline{\Omega}$ da absolyut va tekis yaqinlashuvchi bo`ladi.

Isbot. $\varphi_{1n}=\int_0^p\varphi_{1k}(x)X_n(x)dx$ ni bo`laklab integrallab

$$\varphi_{1n}=\frac{1}{\lambda_n}\varphi_{1n}^{(1)} \qquad (28)$$

ga ega bo`lamiz, bu yerda

$$\varphi_{1n}^{(1)}=\int_0^p\varphi_{1k}'(x)\sqrt{\frac{2}{p}}\cos\lambda_n dx \qquad (29)$$

$$\sum_{k=1}^{\infty} C_k^2 \le \|f\|^2_{L_2(\Omega)}. \qquad (30)$$

ga asosan $\sum_{n=1}^{\infty}\left|\varphi_{1n}^{(1)}\right|^2 \le \|\varphi'_{1k}\|^2_{L_2(0,p)}$. (30) ni quyidagi

$$\sum_{n=1}^{\infty}\left|\varphi_{kn}\right| = \frac{p}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left|\varphi_{kn}^{(1)}\right|,$$

qatorga qo`llab

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|\varphi_{kn}^{(1)}\right| \le \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\left|\varphi_{kn}^{(1)}\right|^2\right)^{1/2} \le \frac{\pi}{\sqrt{6}}\|\varphi'_{1k}\|_{L_2(0,p)}$$ ni topamiz.

Bu yerdan (27) qatorning absolyut va tekis yaqinlashuvchi ekanligi kelib chiqadi. 1 - lemma isbotlandi.

2-lemma. Agar $f_1(x,y)\in C(\overline{\Omega})$, $f_1(0,y)=f_1(p,y)=0$ va $\frac{\partial f_1}{\partial x}\in L_2(\Omega)$ bo`lsa, u holda

$$\sum_{n=1}^{\infty} f_n(y)X_n(x) \qquad (31)$$

qator $\overline{\Omega}$ da absolyut va tekis yaqinlashuvchi bo`ladi.

Isbot. (31) qator uchun $\sum_{n=1}^{\infty}\left|f_n\right|$ qator majoranta bo`ladi. (20) ni bo`laklab integrallab

$$f_n(y) = \frac{1}{\lambda_n} f_n^{(1.0)}(y), \qquad (32)$$

ga ega bo`lamiz, bu yerda

$$f_n^{(1.0)}(y) = \int_0^p \frac{\partial f_1}{\partial x}\sqrt{\frac{2}{p}}\cos\lambda_n x dx .$$

Bunga asosan

$$\sum_{n=1}^{\infty}\left|f_n(y)\right| = \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|f_n^{(1,0)}(y)\right|.$$

Oxirgi qatorga $\sum_{i=1}^{\infty}\left|\xi_i\eta_i\right| \le \left(\sum_{i=1}^{\infty}\left|\xi_i\right|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\infty}\left|\eta_i\right|^2\right)^{\frac{1}{2}}$ tengsizlikni qo`llab

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|f_n^{(1,0)}(y)\right| \le \frac{p}{\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\left|f_n^{(1,0)}(y)\right|^2\right)^{1/2} \le \frac{p}{\sqrt{6}}\left\|\frac{\partial f_1}{\partial x}\right\|_{L_2(\Omega)}.$$

ni topamiz. Bu yerdan (31) qatorning absolyut va tekis yaqinlashuvchi ekanligi kelib chiqadi.

2 – lemma isbotlandi.

Endi (23) – (26) qatorlarning absolyut va tekis yaqinlashuvchi ekanligini isbotlaymiz.

3 - lemma. Agar

$\varphi(x)\in W_2^2(0,p),\ \varphi(0)=\varphi(p)=0,\ \ \psi(x)\in W_2^2(0,p),\ \psi(0)=\psi(p)=0,$

bo`lsa, u holda ixtiyoriy $k\in N$ uchun

$$\sum_{n=1}^{\infty}\frac{e^{\lambda_n(q-y)}-(-1)^k e^{-\lambda_n(q-y)}}{(-1)^k 2sh\lambda_n q}\frac{\varphi_n}{\lambda_n^{k-2}}, \qquad (33)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^k e^{\lambda_n y}-e^{-\lambda_n y}}{(-1)^k 2sh\lambda_n q}\frac{\psi_n}{\lambda_n^{k-2}} \qquad (34)$$

qatorlar $\overline{\Omega}$ da absolyut va tekis yaqinlashuvchidir.

Isbot. Biz quyidagi

$$0\le\frac{e^{\lambda_n(q-y)}-(-1)^k e^{-\lambda_n(q-y)}}{2sh\lambda_n q}\le c_0, \qquad (35)$$

$$0\le\left|\frac{(-1)^k e^{\lambda_n y}-e^{-\lambda_n y}}{(-1)^k 2sh\lambda_n q}\right|\le c_0, \qquad (36)$$

tengsizliklarni isbotlaymiz, bu yerda $c_0=\dfrac{2}{1-e^{-2\frac{\pi q}{p}}}$. Haqiqatdan ham

$$\frac{e^{\lambda_n(q-y)}-(-1)^k e^{-\lambda_n(q-y)}}{e^{\lambda_n q}-e^{-\lambda_n q}}=\frac{e^{\lambda_n(q-y)}}{e^{\lambda_n q}}\frac{1-(-1)^k e^{-2\lambda_n(q-y)}}{1-e^{-2\lambda_n q}}.$$

$$\frac{e^{\lambda_n(q-y)}}{e^{\lambda_n q}}\le 1,\ \ 1-(-1)^k e^{-2\lambda_n(q-y)}\le 2,\ \ 1-e^{-2\lambda_n q}>1-e^{-2\frac{\pi q}{p}},$$

bo`lganligi sababli

$$\frac{e^{\lambda_n(q-y)}-(-1)^k e^{-\lambda_n(q-y)}}{2sh\lambda_n q}\le\frac{2}{1-e^{-2\frac{\pi q}{p}}}=c_0$$

ni olamiz.

(35) tengsizlik isbotlandi. (36) tengsizlikning isboti yuqoridagiga o`xshashdir. $k=2$ bo`lgan holda (33) qator $\overline{\Omega}$ da 1 - lemma sababli absolyut va tekis yaqinlashuvchi bo`ladi. (34) qatorning absolyut va tekis yaqinlashuvchi ekanligining isboti 1 - lemmaning isbotiga o`xshashdir.

4 - lemma. Agar $k>2$ bo`lganda $\dfrac{\partial^{(k-2)}f_1(x,y)}{\partial y^{k-2}}\in C(\overline{\Omega})$ bo`lsa, u holda

$$\sum_{n=1}^{\infty}X_n(x)\frac{e^{\lambda_n(q-y)}-(-1)^k e^{-\lambda_n(q-y)}}{(-1)^k 2sh\lambda_n q}\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\frac{f_n^{(k-2-2s)}(0)}{\lambda_n^{k-2s}}, \qquad (37)$$

$$\sum_{n=1}^{\infty} X_n(x) \frac{(-1)^k e^{\lambda_n y} - e^{-\lambda_n y}}{(-1)^k 2sh\lambda_n q} \sum_{s=0}^{\left[\frac{k-2}{2}\right]} \frac{f_n^{(k-2-2s)}(q)}{\lambda_n^{k-2s}} \qquad (38)$$

qatorlar $\overline{\Omega}$ da absolyut va tekis yaqinlashuvchi bo`ladi.

Isbot. Lemma shartlari tufayli

$$\left| f_n^{(k-2-2s)}(a) \right| \le c_1, \qquad (39)$$

bo`ladi, bu yerda $a=0$ yoki $a=q$, $c_1 = const > 0$. (35) , (36) va (39) tufayli hulosa qilamizki

$$c_0 c_1 \sqrt{\frac{2}{p}} \sum_{n=1}^{\infty} \sum_{s=0}^{\left[\frac{k-2}{2}\right]} \frac{1}{\lambda_n^{k-2s}} \qquad (40)$$

qator (37), (38) qatorlar uchun majoranta bo`ladi. Agar $s=0$ va $k \ge 2$ bo`lsa, u holda (40) qator yaqinlashuvchi bo`ladi. $s = \left[\frac{k-2}{2}\right]$ bo`lganda

$$k-2s = k-2\left[\frac{k-2}{2}\right] = \begin{cases} 3, \; agar\ k - toq\ son\ bo`lsa, \\ 2, \; agar\ k - juft\ son\ bo`lsa \end{cases}$$

ga ega bo`lamiz.

Bundan kelib chiqadiki $s = \left[\frac{k-2}{2}\right]$ bo`lganda, agar $k$-toq son bo`lsa, u holda (40) qator $\sum_{n=1}^{\infty} \lambda_n^{-3}$ ko`rinishga ega, agar $k$-juft son bo`lsa, u holda (40) qator $\sum_{n=1}^{\infty} \lambda_n^{-2}$ ko`rinishga ega bo`ladi. Ikkala holda ham (40) qator yaqinlashuvchi bo`ladi. Shuning uchun (37), (38) qatorlar $\overline{\Omega}$ da absolyut va tekis yaqinlashuvchi bo`ladi.

4 – lemma isbotlandi.

5 - lemma. Agar $m \ge 2$ va $\frac{\partial^{m-1} f_1(x,y)}{\partial y^{m-1}} \in C(\overline{\Omega})$, $\frac{\partial^m f_1(x,y)}{\partial y^m} \in L_2(0,q)$ bo`lsa, u holda

$$\frac{1}{2\lambda_n} \left| \int_0^q e^{-\lambda_n(q-\eta)} f_n^{(m)}(\eta) d\eta \right| \le \frac{C}{n^{3/2}} \left\| f_n^{(m)} \right\|_{L_2(0,q)}, \qquad (41)$$

baholash o`rinlidir, bu yerda $C = \left(\frac{p}{2\pi}\right)^{3/2}$ va $m=2$ yoki $m=k$.

Isbot. $\int_0^1 |x(t) y(t)| dt \le \left(\int_0^1 |x(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^1 |y(t)|^2 dt \right)^{\frac{1}{2}}$ tengsizlikni (41) integralga qo`llab

$$\frac{1}{2\lambda_n}\left|\int_0^q e^{-\lambda_n(q-\eta)} f_n^{(m)}(\eta)d\eta\right| \le \frac{1}{2\lambda_n}\left(\int_0^q e^{-2\lambda_n(q-\eta)}d\eta\right)^{1/2}\left(\int_0^q \left(f_n^{(m)}(\eta)\right)^2 d\eta\right)^{1/2} =$$

$$=\left[\frac{1}{2\lambda_n}\left(1-e^{-2\lambda_n q}\right)\right]^{1/2}\frac{1}{2\lambda_n}\left\|f_n^{(m)}\right\|_{L_2(0,q)} \le \frac{1}{(2\lambda_n)^{3/2}}\left\|f_n^{(m)}\right\|_{L_2(0,q)} =$$

$$=\frac{1}{\left(\frac{2n\pi}{p}\right)^{3/2}}\left\|f_n^{(m)}\right\|_{L_2(0,q)} = \frac{p}{2\pi}\frac{1}{n^{3/2}}\left\|f_n^{(m)}\right\|_{L_2(0,q)} = \frac{C}{n^{3/2}}\left\|f_n^{(m)}\right\|_{L_2(0,q)}.$$

ni olamiz.

5 – lemma isbotlandi.

6 - lemma. Agar $k>2$ va $\frac{\partial^{k-2} f_1(x,y)}{\partial y^{k-2}} \in C(\overline{\Omega})$ bo`lsa, u holda

$$\sum_{n=1}^{\infty} X_n(x)\frac{e^{\lambda_n(q-y)}-(-1)^k e^{-\lambda_n(q-y)}}{(-1)^k 2sh\lambda_n q}\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\frac{f_n^{(k-2-2s)}(0)}{\lambda_n^{k-2-2s}}, \qquad (42)$$

$$\sum_{n=1}^{\infty} X_n(x)\frac{(-1)^k e^{\lambda_n y}-e^{\lambda_n y}}{(-1)^k 2sh\lambda_n q}\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\frac{f_n^{(k-2-2s)}(q)}{\lambda_n^{k-2-2s}} \qquad (43)$$

qatorlar $\overline{\Omega}$ da absolyut va tekis yaqinlashuvchi bo`ladi.

Isbot. (35) va (36) tengsizliklar tufayli hulosa qilamizki

$$c_0\sqrt{\frac{2}{p}}\sum_{n=1}^{\infty}\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\frac{\left|f_n^{(k-2-2s)}(a)\right|}{\lambda_n^{k-2-2s}}, \qquad (44)$$

qator $a=0$ yoki $a=q$ bo`lganda (42) va (43) qatorlar uchun majoranta bo`ladi..

Agar $s=\left[\frac{k-2}{2}\right]$ bo`lsa, u holda

$$k-2-2s = k-2-2\left[\frac{k-2}{2}\right] = \begin{cases} 1, \; agar\; k - toq\; son\; bo`lsa, \\ 0, \; agar\; k - juft\; son\; bo`lsa \end{cases}$$

ga ega bo`lamiz.

Shuning uchun agar k - toq son bo`lsa, u holda $s=\left[\frac{k-2}{2}\right]$ bo`lganda (44) qator

$$\sum_{n=1}^{\infty}\frac{|f_n'(a)|}{\lambda_n}, \qquad (45)$$

ko`rinishga ega bo`ladi, lekin agar k - juft son bo`lsa, u holda (44) qator

$$\sum_{n=1}^{\infty}|f_n(a)|. \qquad (46)$$

ko`rinishda bo`ladi.

$$\sum_{i=1}^{\infty}|\xi_i\eta_i| \le \left(\sum_{i=1}^{\infty}|\xi_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\infty}|\eta_i|^2\right)^{\frac{1}{2}}$$ tengsizlikni (45) qatorga qo`llab

$$\sum_{n=1}^{\infty}\frac{|f_n'(a)|}{\lambda_n} \le \left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}|f_n'(a)|^2\right)^{1/2} \le \frac{p}{\sqrt{6}}\left(\sum_{n=1}^{\infty}|f_n'(a)|^2\right)^{1/2}$$

ni topamiz.

Bu yerdan $\sum_{k=1}^{\infty}C_k^2 = \|f\|^2_{L_2(\Omega)}$ ga asosan

$$\left(\sum_{n=1}^{\infty}|f_n'(a)|^2\right)^{1/2} \le \left\|\frac{\partial f_1(x,a)}{\partial y}\right\|_{L_2(0,q)}$$

ni topamiz.

Bundan (45) qatorning yaqinlashuvchi ekanligi kelib chiqadi. (46) qatorning yaqinlashuvchi ekanligini isbotlash uchun

$$f_n(a) = \int_0^p f_1(x,a)X_n(x)dx$$

integralni bo`laklab integrallaymiz va

$$f_n(a) = \frac{1}{\lambda_n}f_n^{(1)}(a), \qquad (47)$$

ni topamiz, bu yerda

$$f_n^{(1)}(a) = \int_0^p \frac{\partial f_1(x,a)}{\partial x}\sqrt{\frac{2}{p}}cos\lambda_n xdx.$$

(47) ni (46) ga qo`yib

$$\sum_{n=1}^{\infty}|f_n(a)| = \sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|f_n^{(1)}(a)\right|$$

ni topamiz.

Keyinchalik (45) holatdagi kabi

$$\sum_{n=1}^{\infty}|f_n(a)| \le \frac{p}{\sqrt{6}}\left\|\frac{\partial f_1(x,a)}{\partial x}\right\|_{L_2(0,p)}$$

tengsizlikka ega bo`lamiz.

Agar $s < \left[\frac{k-2}{2}\right]$ bo`lsa, u holda (44) qator ixtiyoriy $k > 2$ da yaqinlashuvchi bo`ladi.

Bundan kelib chiqadiki, (42) va (43) qatorlar $\overline{\Omega}$ da absolyut va tekis yaqinlashuvchi bo`ladi.

6 – lemma isbotlandi.

7 - lemma. Agar

$$\frac{\partial^{k-1} f_1(x,y)}{\partial x^i \partial y^j} \in C(\overline{\Omega}), i,j=\overline{0,k-1}, i+j=k-1, \frac{\partial^{2l} f_1}{\partial x^{2l}}(0,y)=\frac{\partial^{2l} f_1}{\partial x^{2l}}(p,y)=0,$$

(48)

bo`lsa, bu yerda

$$l=0,1,\dots,\begin{cases}\frac{k-2}{2}, \text{ agar } k \text{ juft son bo`lsa},\\ \frac{k-3}{2}, \text{ agar } k \text{ toq son bo`lsa},\end{cases}$$

u holda

$$\sum_{n=1}^{\infty}\sum_{s=0}^{\left[\frac{k-2}{2}\right]}\lambda_n^{2s}\left|f_n^{(k-2-2s)}(a)\right|=\sum_{n=1}^{\infty}\left[\left|f_n^{(k-2)}(a)\right|+\lambda_n^2\left|f_n^{(k-4)}(a)\right|+\dots+\right.$$

$$\left.\begin{cases}\lambda_n^{k-2}\left|f_n(a)\right|, \text{ agar } k-\text{ juft son bo`lsa},\\ \lambda_n^{k-3}\left|f_n'(a)\right|, \text{ agar } k-\text{toq son bo`lsa}\end{cases}\right] \quad (49)$$

qator yaqinlashuvchi bo`ladi, bu yerda $a=0$ yoki $a=q$.

Isbot. Faraz qilaylik k- juft son bo`lsin. Agar $s=\left[\frac{k-2}{2}\right]$ bo`lsa, u holda $2s=2\left[\frac{k-2}{2}\right]=k-2$ yaqinlashuvchi bo`ladi. Agar

$$\sum_{n=1}^{\infty}\lambda_n^{k-2}\left|f_n(a)\right| \quad (50)$$

qator yaqinlashuvchi bo`lsa, u holda (49) qator ham yaqinlashuvchi bo`ladi, bu yerda

$$f_n(a)=\int_0^p f_1(x,a)X_n(x)dx.$$

Oxirgi integralni $k-1$ marta bo`laklab integrallab

$$f_n(a)=\frac{1}{\lambda_n^{k-1}}f_n^{(k-1,0)}(a), \quad (51)$$

ga ega bo`lamiz, bu yerda

$$f_n^{(k-1,0)}(a)=\int_0^p\frac{\partial^{k-1}f_1(x,a)}{\partial x^{k-1}}\sqrt{\frac{2}{p}}\sin\left[(k-1)\frac{\pi}{2}+x\right]dx.$$

(51) ga asosan (50) qator

$$\sum_{n=1}^{\infty}\lambda_n^{k-2}\left|f_n(a)\right|\le\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|f_n^{(k-1,0)}(a)\right|$$

ko`rinishga ega bo`ladi.

$\sum_{i=1}^{\infty}|\xi_i\eta_i|\le\left(\sum_{i=1}^{\infty}|\xi_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\infty}|\eta_i|^2\right)^{\frac{1}{2}}$ ni oxirgi tengsizlikning o`ng tomoniga qo`llab

$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left|f_n^{(k-1,0)}(a)\right|\le\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\left|f_n^{(k-1,0)}(a)\right|^2\right)^{1/2}=\frac{p}{\sqrt{6}}\left(\sum_{n=1}^{\infty}\left|f_n^{(k-1,0)}(a)\right|^2\right)^{1/2}$ ni topamiz.

$\sum_{k=1}^{\infty}C_k^2=\|f\|_{L_2(\Omega)}^2$ ni qo`llab

$\left(\sum_{n=1}^{\infty}\left|f_n^{(k-1,0)}(a)\right|^2\right)^{1/2}\le\left\|\frac{\partial^{k-1}f_1(x,a)}{\partial x^{k-1}}\right\|_{L_2(0,p)}$ ni topamiz.

Bundan kelib chiqadiki,

$$\sum_{n=1}^{\infty}\lambda_n^{k-2}|f_n(a)|\le\frac{p}{\sqrt{6}}\left\|\frac{\partial^{k-1}f_1(x,a)}{\partial x^{k-1}}\right\|_{L_2(0,p)}$$

bo`ladi.

(50) qatorning yaqinlashuvchi ekanligi isbotlandi.

Faraz qilaylik k - toq son va $s=\left[\frac{k-2}{2}\right]$ bo`lsin. U holda $2s=2\left[\frac{k-2}{2}\right]=k-3$ bo`ladi. Agar $\sum_{n=1}^{\infty}\lambda_n^{k-3}\left|f_n'(a)\right|$ qator yaqinlashuvchi bo`lsa, u holda (49) qator ham $k$ - toq son uchun yaqinlashuvchi bo`ladi. Bu tasdiqning isboti (50) qator yaqinlashuvchi ekanligi isboti kabi isbotlanadi.

7 – lemma isbotlandi.

8 - lemma. Faraz qilaylik 7 – lemmaning shartlari o`rinli bo`lsin, u holda

$$\sum_{n=1}^{\infty}\sum_{s=0}^{k-1}\lambda_n^{k-2-s}\left|f_n^{(s)}(a)\right|,\ a=0,q$$

qator yaqinlashuvchi bo`ladi va

$$\sum_{n=1}^{\infty}\sum_{s=0}^{k-1}\lambda_n^{k-2-s}\left|f_n^{(s)}(y)-f_n^{(s)}(0)e^{-\lambda_n y}\right|,$$

$$\sum_{n=1}^{\infty}\sum_{s=0}^{k-1}\lambda_n^{k-2-s}\left|f_n^{(s)}(y)-f_n^{(s)}(q)e^{-\lambda_n(q-y)}\right|$$

qatorlar $\overline{\Omega}$ da absolyut va tekis yaqinlashuvchi bo`ladi.

Isboti 7 – lemmaning isbotiga o`xshash.

Isbotlangan lemmalar natijalaridan foydalanib quyidagi teoremani isbotlaymiz.

2 - teorema. Faraz qilaylik

$\varphi(x)\in W_2^2(0,p),\varphi(0)=\varphi(p)=0,\psi(x)\in W_2^2(0,p),\psi(0)=\psi(p)=0,$

$$\frac{\partial^{k-1} f_1(x,y)}{\partial x^i \partial y^j} \in C(\overline{\Omega}), i+j=k-1, i,j=\overline{0,k-1}, \frac{\partial^k f_1(x,y)}{\partial y^k} \in L_2(\Omega),$$

$$\frac{\partial^{2l} f_1}{\partial x^{2l}}(0,y) = \frac{\partial^{2l} f_1}{\partial x^{2l}}(p,y) = 0, \quad l = 0,1,\ldots, \begin{cases} \frac{k-2}{2}, \ agar\ k - juft\ son\ bo`lsa, \\ \frac{k-3}{2}, \ agar\ k - toq\ son\ bo`lsa, \end{cases}$$

bo`lsin, u holda (23)-(25) qatorlar $\overline{\Omega}$ da absolyut va tekis yaqinlashuvchi bo`ladi, (23) ning yechimi Ω da (11) tenglamani va (12)-(13) shartlarni qanoatlantiradi.

Isbot. (24) va (25) ni qo`shib (23) ning yechimi Ω da (11) tenglamani qanoatlantirishiga oson ishonch hosil qilamiz. $X_n(x)$ funksiyaning hossalaridan kelib chiqadiki (23) tenglamaning yechimlari (12) va (13) shartlarni qanoatlantiradi. (23)-(25) qatorlarning $\overline{\Omega}$ da absolyut va tekis yaqinlashuvchi ekanligi 3-5 lemmalardan kelib chiqadi, shuning uchun $u, u_{xx}, u_{yy} \in C(\overline{\Omega})$. Bundan hulosa qilish mumkinki $u \in C_{x,y}^{2,2}(\overline{\Omega})$. (26) qatorning $\overline{\Omega}$ da absolyut va tekis yaqinlashuvchi ekanligi 5 va 7 lemmalardan kelib chiqadi. Bundan $\frac{\partial^k u}{\partial y^k} \in C(\overline{\Omega})$ kelib chiqadi. (26) da $y \to 0$ va $y \to q$ da limitga o`tib (23) ning yechimi (4) va (5) shartlarni qanoatlantirishiga ishonch hosil qilamiz.

Teorema isbotlandi.

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