IKKINCHI TARTIBLI XUSUSIY HOSILALI BUZILADIGAN DIFFERENSIAL TENGLAMALAR UCHUN CHEGARAVIY MASALA

Qodiraliyev Asadbek Ikromali o‘g‘li

Annatotsiya. *Ushbu maqolada buziladigan ikkinchi tartibli tenglama uchun boshlang`ich-chegaraviy malasa bayon qilingan va tadqiq etilgan. Malasaning yechimining yagonali energiya integrallari usulidan foydalanib isbotlangan. Malasa yechimining mavjud ekanligi esa o`zgaruvchilarni ajratish usuli yordamida ko`rsatilgan.*

Kalit so‘zlar: *buziladigan differensial tenglamalar, chegaraviy masala, energiya integrallari usuli, o`zgaruvchilarni ajratish usuli.*

Annotation. *In this article, the initial-boundary problem for the degenerative second-order equation is described and researched. The solution of the problem is proved using the method of energy integrals. The existence of a solution to the problem is shown using the method of separation of variables.*

Key words: *Degenerative differential equations, boundary value problem, method of energy integrals, method of separation of variables.*

Аннотация. *В данной статье описана и исследована начально-краевая задача для вырождающегося уравнения второго порядка. Решение задачи доказывается методом интегралов энергии. Существование решения задачи показано с помощью метода разделения переменных.*

Ключевые слова: *Вырождающиеся дифференциальные уравнения, краевая задача, метод интегралов энергии, метод разделения переменных.*

I. Kirish. Masalaning qo`yilishi. O‘tgan asrdan boshlab buziladigan ikkinchi tartibli xususiy hosilali differensial tenglamalar uchun chegaraviy masalalar o‘rganish boshlangan va anchagina boy tarixga ega. Bu tadqiqotlar ko‘plab olib borilishiga qaramasdan so‘ngi yillarda ham giperbolik tipdagi buziladigan ikkinchi tur tenglamalar uchun boshlang‘ich va chegaraviy masalalar o‘rganib kelinmoqda(masalan [1]-[9]).

So‘ngi yillarda esa kasr tartibli to‘lqin tenglamalari uchun chegaraviy masalalarni o‘rganishga bo‘lgan qiziqish ortgan. Shu sababli biz ushbu ishda ikkinchi tartibli xususiy hosilali buziladigan differensial tenglamalar uchun chegaraviy masalani ko‘rib chiqamiz.

$\Omega=\{(x,t):0<x<1,0<t<T\}$ sohada

$$ {}_C D_{0t}^{\alpha} u(x,t)=\left[x^{\beta}u_x(x,t)\right]_x \qquad (1)$$

tenglamani qaraylik, bu erda ${}_C D_{0t}^{\alpha}$ - Капуто ma’nosidagi kasr tartibili operator [7]

$$ {}_C D_{0t}^{\alpha} u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_z(x,z)}{(t-z)^{\alpha}}dz ,$$

α,β,T - o'zgarmas haqiqiy sonlar bo'lib, $0\le\beta<1$, $0<\alpha<1$, $T>0$.

A masala. Shunday $u(x,t)$ topilsinki u quydagi xossalarga ega bo'lsin:

1) $u(x,t)$, $x^{\beta}u_x(x,t)\in C(\overline{\Omega})$, $\left[x^{\beta}u_x(x,t)\right]_x$, ${}_C D_{0t}^{\alpha}u(x,t)\in C(\Omega)$ sinfga tegishli;

2) Ω soha (1) tenglamani qanoatlantiradi.

3) Ω soha chegarasida esa ushbu

$$u(x,0)=\varphi(x),\ x\in[0,1], \tag{2}$$

$$u(0,t)=0,\ u(1,t)=0,\ t\in[0,T]; \tag{3}$$

boshlang`ich va chegaraviy shartlarni qanoatlantiradi, bu erda $\varphi(x)$ - berilgan funksiya.

II. Masalaning tadqiqoti.

Dastalab, masalaning yechimini yagona ekanligini ko'rsatish maqsadida quyidagi yordamchi lemmani keltiramiz.

1-lemma. *Agar $V(x,0)=0$ va $\gamma\in(0,1)$ bo'lsa, u xolda quydagi tengsizlik o'rinli:*

$$I(x)=\int_0^T V(x,t)\,{}_C D_{0t}^{\gamma}V(x,t)dt\ge 0$$

1-teorema. A masala bittadan ortiq echimga ega emas.

Isbot. Faraz qilaylik, $u_1(x,t)$ va $u_2(x,t)$ echimlarga ega bo'lsin. Undan $u(x,t)=u_1(x,t)-u_2(x,t)$ funksiya Ω sohada (1) tenglamaga mos bir jinsli tenglamani, uning chegarasida esa $u(x,0)=0$, $u(0,t)=0$, $u(1,t)=0$, tengliklarni qanoatlantiradi.

(1) tenglamani $u(x,t)$ funksiyaga ko'paytirib, Ω soha bo'yicha integrallaymiz:

$$\int_0^1\int_0^T u(x,t)\,{}_C D_{0t}^{\alpha}u(x,t)dxdt=\int_0^T\int_0^1\left[x^{\beta}u_x(x,t)\right]_x u(x,t)dxdt.$$

Tenglikni o'ng tomonidagi integralni bo'laklab integrallab, $u(0,t)=0$, $u(1,t)=0$ tengliklardan foydalanib,

$$\int_0^T\int_0^1 x^{\beta}u_x^2(x,t)dxdt=-\int_0^1\int_0^T u(x,t)\,{}_C D_{0t}^{\alpha}u(x,t)dxdt$$

tenglikni hosil qilamiz.

1- lemmadan foydalanib,

$$\int_0^T\int_0^1 x^{\beta}u_x^2(x,t)dxdt=0$$

tenglikni hosil qilamiz. Bundan esa $x^{\beta}u_x^2(x,t)=0$, $(x,t)\in\Omega$ $u(x,t)=\tau(t)$, $(x,t)\in\Omega$ ekanligi kelib chiqadi. $u(x,t)\in C(\overline{\Omega})$ va $u(x,0)=0$, $x\in[0,1]$ ekanligini etiborga olsak $\forall(x,t)\in\overline{\Omega}$ uchun $u(x,t)\equiv 0$, ya`ni $u_1(x,t)\equiv u_2(x,t)$ ekanligi kelib chiqadi 1-teorema isbotlandi.

Endi masla echimini mavjudligini ko'rsatish maqsadida (1) tenglamaning echimini

$$u(x,t)=X(x)T(t) \qquad (4)$$

ko'rinishda qidirib, uni (1) tenglamaga qo'yib, ba'zi soddalashtirishlarni amalga oshirib,

$$\frac{{}_C D_{0t}^{\alpha} T(t)}{T(t)} = \frac{\left[x^{\beta} X'(x)\right]'}{X(x)} \qquad (5)$$

tenglamani hosil qilamiz.

(5) tenglamani uning o'ng tomoni t ga, chap tomoni esa x ga bog'liq bo'lmagani uchun uni o'zgarmas $(-\lambda)$ soniga tenglab,

$${}_C D_{0t}^{\alpha} T(t) + \lambda T(t) = 0,\ 0<t<T; \qquad (6)$$

$$\left[x^{\beta} X'(x)\right]' + \lambda X(x) = 0,\ 0<x<1; \qquad (7)$$

ko'rinishidagi oddiy differensial tenglamalarni hosil qilamiz. (5) tenglikka asosan (3) shartlardan $X(x)$ funksiya uchun

$$X(0)=0,\ X(1)=0 \qquad (8)$$

shartlar kelib chiqadi.

$\{(7),(8)\}$ spektral masalaning xos sonlarini topish uchun (7) tenglamani $X(x)$ ga ko'paytirib $[0,1]$ oraliqda integrallaymiz:

$$\int_0^1 \left[x^{\beta} X'(x)\right]_x X(x)dx + \lambda \int_0^1 X^2(x)dx = 0. \qquad (9)$$

(9) ni bir marta bo'laklab integrallab, (8) shartni e'tiborga olsak, u holda

$$\int_0^1 x^{\beta}\left[X'(x)\right]^2 dx = \lambda \int_0^1 X^2(x)dx \qquad (10)$$

tenglikni hosil qilamiz, undan esa $\lambda \geq 0$ ekanligi kelib chiqadi.

Daastlab, $\lambda=0$ bo'lsin (8) ga ko'ra $\left[x^{\beta} X'(x)\right]'=0$ bo'lishidan $X(x)=C_1\dfrac{x^{1-\beta}}{1-\beta}+C_2$ kelib chiqadi, uni (8) shartga bo'ysundirsak, $X(x)\equiv 0$ ekanligi kelib chiqadi. Demak $\lambda=0$ xos son emas.

$\lambda>0$ bo'lsin. U holda X(x)=V(z)

$$z=\frac{x^{2-\beta}}{(2-\beta)^2} \qquad (11)$$

belgilash kiritib olamiz. (11) dan kerakli hosilalarni olib, (7) ga qo'yib, bazi soddalashtirishlarni amalga oshirib,

$$zV''(z)+\frac{V'(z)}{2-\beta}+\lambda V(z)=0 \qquad (12)$$

tenglamani hosil qilamiz.

(12) tenglama Bessel tenglamasiga keltrilgan tenglama bo‘lib, uning echimi

$$V(z)=\left(2\sqrt{z}\right)^{\frac{1-\beta}{2-\beta}}\left[C_1J_v\left(2\sqrt{z\lambda}\right)+C_2J_{-v}\left(2\sqrt{z\lambda}\right)\right]$$

ko‘rinishda bo‘ladi.

(11) belgilashga ko‘ra (8) tenglamaning umumiy echimi

$$X(x)=\left(\frac{2}{2-\beta}\right)^{\frac{1-\beta}{2-\beta}}x^{\frac{1-\beta}{2}}\left[C_1J_v\left(2\sqrt{\lambda}\,\frac{x^{\frac{2-\beta}{2}}}{2-\beta}\right)+C_2J_{-v}\left(2\sqrt{\lambda}\,\frac{x^{\frac{2-\beta}{2}}}{2-\beta}\right)\right] \qquad (13)$$

ko‘rinishda bo‘ladi bu erda C_1, C_2, v — o‘zgarmas sonlar, $v=(1-\beta)/(2-\beta)$.

Endi (13) echimni (8) chegaraviy shartlarni birinchisiga bo‘ysindirib, $C_2=0$ ekanligini topamiz.

$X(1)=0$ dan esa $C_1J_v\left(\frac{2\sqrt{\lambda}}{2-\beta}\right)=0$ tenglamani hosil qilamiz. $C_1\neq 0$ deb, $J_v\left(\frac{2\sqrt{\lambda}}{2-\beta}\right)=0$ tenglamani echamiz. $v>0$ bo‘lganligi uchun $J_v\left(\frac{2\sqrt{\lambda}}{2-\beta}\right)=0$ tenglama absalyut qiymati bo‘yicha cheksiz kattalashib boruvchi sanoqli sondagi xaqiqiy echimlarga ega [8]. Uning $n-$musbat echimini θ_n bilan belgilasak $\{(7),(8)\}$ masalaning sanoqli sondagi

$$\lambda_n=\left[\frac{2-\beta}{2}\theta_n\right]^2,\ n=1,2,...$$

xos qiymatlariga ega bo‘lamiz, unga mos keluvchi xos funksiyalar esa

$$X_n(x)=x^{\frac{1-\beta}{2}}J_v\left[\theta_n x^{\frac{2-\beta}{2}}\right],\ n=1,2,... \qquad (14)$$

ko‘rinishida bo‘ladi.

2-lemma. *(14) formula bilan aniqlanuvchi $X_n(x)$, $n\in N$ funksiyalar $(0,1)$ kesmada ortonormal va to‘la sistema tashkil qiladi.* [10]

Yuqoridagilarga asosan A masalaning formal yechimi

$$u(x,t)=\sum_{n=1}^{+\infty}x^{\frac{1-\beta}{2}}J_v\left[\theta_n x^{\frac{2-\beta}{2}}\right]\varphi_nE_{\alpha,1}\left(-\lambda_n t^{\alpha}\right) \qquad (15)$$

ko‘rinishda bo‘ladi, bu erda $v=(1-\beta)/(2-\beta)$, $\varphi_n=\dfrac{1}{\mu_n}\int_0^1\varphi(x)X_n(x)dx$,

$\mu_n=\int\limits_0^1 X_n^2(x)dx=\left[1/(2-\beta)\right]J_{v+1}^2(\theta_n)$, $\theta_n=\dfrac{2}{2-\beta}\sqrt{\lambda_n}$, $E_{\alpha,\beta}(z):=\sum\limits_{k=1}^{\infty}\dfrac{z^k}{\Gamma(\alpha k+\beta)}$,

$(z,\beta\in J\ ;\Re(\alpha)>0)$ - Mittag-Leffer funksiyasi.

2-Teorema. *Agar* $\varphi(x)$, $x^\beta\varphi'(x)$, $\left[x^\beta\varphi'(x)\right]'$, $\varphi(0)=\varphi(1)=0$ *va* $\left[\varphi'(x)x^\beta\right]'_{x=0}=0$, $\left[\varphi'(x)x^\beta\right]'_{x=1}=0$ *bo‘lsa,* (15) *qator bilan aniqlangan* $u(x,t)$ *funksiya A masalaning echimi bo‘ladi.*

Isbot. Teoremani isbotlash uchun (15) va $x^\beta u_x(x,t)$, $\left[x^\beta u_x(x,t)\right]_x$, ${}_C D_{0t}^{\alpha}u(x,t)$ ga mos keluvchi qatorlarni tekis yaqinlashuvchi ekanligini isboltash etarli.

Dastlab (15) qatorning tekis yaqinlashishini ko‘rsatish maqsadida qatorni baholaymiz:

$$\left|u(x,t)\right|\le\sum_{n=0}^{\infty}\left|X_n(x)\right|\left|T_n(t)\right|=\sum_{n=0}^{\infty}\left|x^{\frac{1-\beta}{2}}J_v\left(\theta_n x^{\frac{2-\beta}{2}}\right)\right|\left|\varphi_n E_\alpha\left(-\lambda_n t^\alpha\right)\right|. \qquad (16)$$

buning uchun dastlab Bessel-Klifford funksiyasidan foydalanib,

$$X_n(x)=C_3\left(\sqrt{\lambda_n}\right)^{\frac{1-\beta}{2-\beta}}x^{1-\beta}\overline{J_v}\left(\frac{2\sqrt{\lambda_n}}{2-\beta}x^{\frac{2-\beta}{2}}\right) \qquad (17)$$

ko‘rinishda yozib olamiz, bu erda

$$\overline{J_v}\left(\theta_n x^{\frac{2-\beta}{2}}\right)=\left(\frac{\theta_n}{2}\right)^{-v}\Gamma(v+1)x^{\frac{\beta-1}{2}}J_v\left(\theta_n x^{\frac{2-\beta}{2}}\right)$$

$\left|\overline{J_v}(z)\right|\le 1$ va $\left|E_\alpha(-z)\right|<C$ tengizliklardan foydalanib,

$$\left|u(x,t)\right|\le C_4\sum_{n=1}^{\infty}\left|\varphi_n\right|\left|\sqrt{\lambda_n}\right|^{(1-\beta)/(2-\beta)}$$

tengsizlikni hosil qilamiz.

Koshi-Bunyakoviskiy tengsizligini qo‘llab,

$$\left|u(x,t)\right|\le C_4\left(\sum_{n=0}^{+\infty}\lambda_n\varphi_n^2\sum_{n=0}^{+\infty}\lambda_n^{(-2)/(2-\beta)}\right)^{1/2} \qquad (18)$$

tengsizlikni hosil qilamiz.

Endi ushbu tenglikni qaraymiz:

$$\varphi_n=\frac{2-\beta}{\overline{J_{v+1}^2}(\theta_n)}\int\limits_0^1\varphi(x)X_n(x)dx. \qquad (19)$$

(19) ifodani bo‘laklab integrallab

$$\sqrt{\lambda_n}\varphi_n = \frac{2-\beta}{J_{v+1}^2(\theta_n)}\left(\frac{\varphi(x)X'(x)x^\beta}{\sqrt{\lambda_n}}\Bigg|_0^1 - \int_0^1 \varphi'(x)x^{\frac{\beta}{2}}\left(\frac{X'(x)x^{\frac{\beta}{2}}}{\sqrt{\lambda_n}}\right)dx\right)$$

tenglikni hosil qilamiz, bu erda $\varphi(1)=0$, $\varphi(0)=0$ bo'lganligi uchun

$$\sqrt{\lambda_n}\varphi_n = \frac{\beta-2}{J_{v+1}^2(\theta_n)}\int_0^1 \varphi'(x)x^{\frac{\beta}{2}}dx$$

$\varphi'(x)\in L_2(0,1)$ ekanligdan oxirigi ifoda Furye koeffitsenti bo'ladi, u holda Bessel tensizligiga ko'ra

$$\sum_{n=0}^{\infty}\left|\lambda_n\varphi_n{}^2\right| \le C_4\int_0^1 (\varphi'(x))^2 x^\beta dx \le M \qquad (20)$$

ifodani hosil qilamiz, bu erda $C_4 = \left(\frac{\beta-2}{J_{v+1}^2(\theta_n)}\right)^2$.

(20) ga ko'ra (18) tengsizlikdagi birinchi qator yaqinlashuvchi, $\beta\in(0,1)$ bo'lgani uchun $\sum_{n=0}^{\infty}\lambda_n^{\frac{-2}{2-\beta}}$ qator umumlashgan garmo'nik bo'lib u yaqinlashuvchi bo'ladi.

Bularga ko'ra (18) yoki (15) qator tekis yaqinlashuvchi bo'lishi kelib chiqadi.

Endi $\left[x^\beta u_x(x,t)\right]_x$ funksiya mos qatorni tekis yaqinlashuvchi ekanligini ko'rsatish maqsadida

$$\left[x^\beta u_x(x,t)\right]_x = \sum_{n=1}^{\infty}\left[x^\beta X_n'(x)\right]'\varphi_n E_\alpha(-\lambda_n t^a)$$

tenglikni qaraymiz. Bu tenglikda (7) tenglamadan foydalanib,

$$\left[x^\beta u_x(x,t)\right]_x = \sum_{n=1}^{\infty}-\lambda_n X_n(x)\varphi_n E_\alpha(-\lambda_n t^a) \qquad (20)$$

tenglikni hosil qilamiz.

Yuqoridagi tengsizliklarga asoslanib,

$$\left|\left[x^\beta u_x(x,t)\right]_x\right| \le C_4\sum_{n=0}^{+\infty}|\lambda_n\varphi_n|\left|\sqrt{\lambda_n}^{(1-\beta)/(2-\beta)}\right|$$

tengsizlikni hosil qilamiz.

Oxirigi tengsizlikka Koshi -Bunyakoviskiy tengsizligini qo'lab,

$$\left|x^\beta u_x(x,t)\right| \le C_4\left(\sum_{n=0}^{+\infty}\left|\lambda_n^3\varphi_n^2\right|\sum_{n=0}^{+\infty}\left|\sqrt{\lambda_n}^{(-2)/(2-\beta)}\right|\right)^{1/2} \qquad (21)$$

tengsizlikni hosil qilamiz.

Endi (19) ifodani bo'laklab integrallab

$$\lambda_n\sqrt{\lambda_n}\varphi_n=\frac{(2-\beta)}{J_{\nu+1}^2(\theta_n)}\left(\frac{\left[\varphi'(x)x^\beta\right]' x^\beta X_n'(x)}{\sqrt{\lambda_n}}\Bigg|_{x=0}^{x=1}-\int_0^1 x^{\frac{\beta}{2}}\left[\varphi'(x)x^\beta\right]''\left(\frac{x^{\frac{\beta}{2}}X_n'(x)}{\sqrt{\lambda_n}}\right)dx\right)$$

tenglikni hosil qilamiz, bu erda $\left[\varphi'(x)x^\beta\right]'_{x=0}=0$, $\left[\varphi'(x)x^\beta\right]'_{x=1}=0$ bo'lganligi uchun

$$\lambda_n\sqrt{\lambda_n}\varphi_n=\frac{\beta-2}{J_{\nu+1}^2(\theta_n)}\int_0^1 x^{\frac{\beta}{2}}\left[\varphi'(x)x^\beta\right]''dx$$

$\left[\varphi'(x)x^\beta\right]''\in L_2(0,1)$ ekanligdan oxirigi ifoda Furye koeffitsenti bo'ladi, u holda Bessel tensizligiga ko'ra

$$\sum_{n=1}^{\infty}\lambda_n^3\varphi_n^2\le C_5\int_0^1\left(\left[\tau'(x)x^\beta\right]''\right)^2 x^\beta dx\le M \qquad (22)$$

ifodani hosil qilamiz, bu erda $C_5=\left(\frac{\beta-2}{J_{\nu+1}^2(\theta_n)}\right)^2$

(22) ga ko'ra (21) tengsizlikdagi birinchi qator yaqinlashuvchi, $\beta\in(0,1)$ bo'lgani uchun $\sum_{n=0}^{\infty}\lambda_n^{\frac{-2}{2-\beta}}$ qator umumlashgan garmo'nik bo'lib u yaqinlashuvchi bo'ladi.

Bularga ko'ra (22) yoki (20) qator tekis yaqinlashuvchi bo'lishi kelib chiqadi.

Teorema isbotlandi.

FOYDALANILGAN ADABIYOTLAR:

1. Терсенов С.А.О задаче Коши с данными на линии вырождения типа для гиперболического уровнения // Диффер. Уравн., 1966. Т.2, №1. С 125-130.

2.Терсенов С.А К теории гиперболических уровнений с данными на линии вырождения типа //Сиб. Матем. Журн., 1961. Т.2, № 6 С. 913-935.

3. Терсенов С. А. Введение в теорию уровнений, вырождающихся на границе. Новосибрск: : НГУ, 1937. 144 с.

4.Смирнов М.М. Вырождающиеся гиперболические уравнения. Минск: Высш.шк., 1977. 157 с.

5.Хайруллин Р. С. Задача Трикоми для уравнения второго рода с сильным вырождением. Казань: Казанск. унив., 2015. 336 с. END: UWLDMB.

6.Мамадалиев Н. К.О представлении решения видоизмененной задачи Коши // Сиб. матем. журн., 2000. Т. 41, №5. С. 1087-1097.

7. Уринов А. К., Окбоев А. Б. Видоизмененная задача Коши для одного вырождающе гося гиперболического уравнения второго рода // Укр. матем. журн., 2020.Т. 72, № 1 С. 100-118

8. Urinov A. K. Okboev A. B. On a Cauchy type problem for a second kind degenerating hyperbolic equation // Lobachevskit J. Math., 2022. vol. 43, no. 3. pp. 793-803. EDN: QPEVQB. DOI: https://doi.org/10.1134/S199508022206032.

9. Уринов А. К., Усмонов Д. А. О видоизменной задаче Коши для одного вырождаю щегося гиперболического уравнения второго рода // Бюл. Инст . mam ., 2021. Т.4, № 1. С. 46-63

10. Ватсон Г.Н., Теория бесселевых функций.Часть первая. -М.: Изд-во ИЛ,1949,789 с.

11. Usmonov D. A. Problem with shift condition for a second kind degenerated equation of hyperbolic type // Scientific Jurnal of the Fergana State University. 2020. № 6. pp.6-10.

12. D.A. Usmonov. A Cauchy-Goursat problem for a second kind degenerated equation of hyperbolic type // Scientific Bulletin. Physical and mathematical Research Vol.3 Iss.1 2021. pp. 76-83

13. O'rinov A.Q. Usmonov D.A. A Cauchy-Goursat problem for a second kind degenerated equation of hyperbolic type // Scientific Jurnal of the Fergana State University.2021/ №5. pp. 6-16

14. O'rinov A.Q. Usmonov D.A. A mixed problem for Secondary of the hyperbolic type with two fault lines the type equation // Scientific Bullettin of NamSU. 2022 / №5. Pp. 119-129

15. Уринов А. К., Усмонов Д. А. Initial boundary value problem for a hyperbolic equation with three lines of degeneracy of the second kind // Scientific Bulletin. Physical and mathematical Research Vol.4 Iss.1 2022. Pp. 56-59