DIFFERENSIAL TENGLAMALAR SISTEMASI UCHUN KOSHI MASALASI

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Annotatsiya. *Ushbu maqolada kasr tartibli diffrerensial operatorlarni o'z ichiga oluvchi o'zgarmas koeffitsiyentli bir jinsli bo'lmagan differensial tenglamalar sistemasi uchun qo'yilgan Koshi masalasining yechimi Dalamber usulidan foydalanib topilgan. Bunda qo'yilgan masala ikkinchi tur Volterra integral tenglamasiga ekvivalent keltirib, integral tenglamaning yechimi ketma-ket yaqinlashish usulidan foydalanib topilgan.*

Kalit so'zlar: *differensial tenglamalar sistemasi, Koshi masalasi, kasr tartibli differensial operatorlar, Dalamber usuli.*

Mazkur maqola [1], [2] ishlarning davomi bo'lib, xususiy hollarda mazkur maqolada olinadigan natijalardan [1], [2] ishlarda olingan natijalar kelib chiqadi.

Quyidagi

(1) $$\begin{cases} D_{mt}^{\alpha}x(t)=aD_{mt}^{\beta}x(t)+bD_{mt}^{\beta}y(t)+f_1(t), \\ D_{mt}^{\alpha}y(t)=cD_{mt}^{\beta}x(t)+dD_{mt}^{\beta}y(t)+f_2(t), \end{cases} \quad t>m$$

sistemaning

$$\lim_{t\to+m} I_{mt}^{1-\alpha}x(t)=x_0, \ \lim_{t\to+m} I_{mt}^{2-\alpha}x(t)=x_1,$$

(2) $$\lim_{t\to+m} I_{mt}^{1-\alpha}y(t)=y_0, \ \lim_{t\to+m} I_{mt}^{2-\alpha}y(t)=y_1,$$

shartlarni qanoatlantiruvchi yechimini topish masalasini qaraylik, bu yerda D_{mt}^{γ} va I_{mt}^{γ} kasr tartibli Riman-Liuvill ma'nosidagi differensial va integral operatorlar [3]:

$$D_{mt}^{\alpha}f(t)=\frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_m^t (t-z)^{1-\alpha} f(z)dz,$$

$$I_{mt}^{1-\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_m^t (t-z)^{-\alpha} f(z)dz,$$

$$I_{mt}^{2-\alpha}f(t)=\frac{1}{\Gamma(2-\alpha)}\int_m^t (t-z)^{1-\alpha} f(z)dz,$$

$$D_{mt}^{\beta}f(t)=\frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_m^t (t-z)^{-\beta} f(z)dz,$$

$\Gamma(g)$–Eylerning gamma funksiyasi [4], $\alpha,\beta,a,b,c,d,x_0,x_1,y_0,y_{1,}$ –berilgan haqiqiy sonlar bo'lib, $1<\alpha<2,\ 0<\beta<1;$ $f_1(t)$ va $f_2(t)$–berilgan funksiyalar, $x(t)$ va $y(t)$lar esa noma'lum funksiyalar.

$\{(1),(2)\}$ Koshi masalasining yechimini topish bilan shug‘ullanamiz. Shu maqsadda (1) ning ikkinchi tenglamasini λ songa ko‘paytirib so‘ngra birinchi tenglamaga qo‘shamiz. Natijada yig‘indini differensiallash formulasini e’tiborga olgan holda ushbu

(3) $$D_{mt}^{\alpha}\left[x(t)+\lambda y(t)\right]=(a+\lambda c)D_{mt}^{\beta}x(t)+(b+\lambda d)D_{mt}^{\beta}y(t)+f_1(t)+\lambda f_2(t)$$

tenglikni hosil qilamiz.

(3) tenglikning o‘ng tomonidagi birinchi hadni qavsdan tashqariga chiqarib

(4) $$D_{mt}^{\alpha}\left[x(t)+\lambda y(t)\right]=(a+\lambda c)\left(D_{mt}^{\beta}x(t)+\frac{b+\lambda d}{a+\lambda c}D_{mt}^{\beta}y(t)\right)+f_1(t)+\lambda f_2(t)$$

tenglikni hosil qilamiz.

λ sonni shunday tanlaylikki, u ushbu $\frac{b+\lambda d}{a+\lambda c}=\lambda$ tenglamaning yechimi bo‘lsin. Faraz qilaylik, λ_1 va λ_2 oxirgi tenglamaning ildizlari bo‘lib, $\lambda_1\neq\lambda_2$ bo‘lsin. U holda bu sonlarni (4)ga qo‘yib, differensiallashning sodda qoidalarini hisobga olgan holda ushbu

(5) $$D_{mt}^{\alpha}\left[x(t)+\lambda_i y(t)\right]=(a+\lambda_i c)D_{mt}^{\beta}\left[x(t)+\lambda_i y(t)\right]+f_1(t)+\lambda f_2(t),\ i=1,2$$

ko‘rinishdagi $x(t)+\lambda_i y(t)$ noma’lumlarga nisbatan chiziqli ikkita tenglamaga ega bo‘lamiz. Soddalik maqsadida quyidagi belgilashlarni kiritaylik:

(6) $$x(t)+\lambda_i y(t)=z(t),\quad f_1(t)+\lambda f_2(t)=f_{3,i}(t),\quad a+\lambda_i c=\tilde{\lambda}_i,\quad i=1,2.$$

U holda (5) tenglama ushbu ko‘rinishga keladi:

(7) $$D_{mt}^{\alpha}z(t)=\tilde{\lambda}_i D_{mt}^{\beta}z(t)+f_{3,i}(t),\quad i=1,2.$$

Ushbu

$$I_{mt}^{\alpha}D_{mt}^{\alpha}z(t)=z(t)-\frac{(t-m)^{\alpha-1}}{\Gamma(\alpha)}I_{mt}^{1-\alpha}z(m)-\frac{(t-m)^{\alpha-2}}{\Gamma(\alpha-1)}I_{mt}^{2-\alpha}z(m),$$

(8) $$I_{mt}^{\alpha}D_{mt}^{\beta}z(t)=I_{mt}^{\alpha-\beta}z(t)-\frac{(t-m)^{\alpha-1}}{\Gamma(\alpha)}I_{mt}^{1-\beta}z(m)$$

formulalarni e’tiborga olgan holda (7) tenglikning har ikki tomoniga I_{mt}^{α} operatorni tatbiq qilamiz:

$$z(t)-\frac{(t-m)^{\alpha-1}}{\Gamma(\alpha)}I_{mt}^{1-\alpha}z(m)-\frac{(t-m)^{\alpha-2}}{\Gamma(\alpha-1)}I_{mt}^{2-\alpha}z(m)=$$

$$=\tilde{\lambda}_i I_{mt}^{\alpha-\beta}z(t)-\tilde{\lambda}_i\frac{(t-m)^{\alpha-1}}{\Gamma(\alpha)}I_{mt}^{1-\beta}z(m)+I_{mt}^{\alpha}f_{3,i}(t).$$

Natijada (2) shartlarni e’tiborga olsak, z(t) noma’lum funksiyaga nisbatan

(9) $$z(t)-\tilde{\lambda}_i I_{mt}^{\alpha-\beta}z(t)=\frac{(t-m)^{\alpha-1}}{\Gamma(\alpha)}I_{mt}^{1-\alpha}z(m)+\frac{(t-m)^{\alpha-2}}{\Gamma(\alpha-1)}I_{mt}^{2-\alpha}z(m)-$$

$$-\tilde{\lambda}_i\frac{(t-m)^{\alpha-1}}{\Gamma(\alpha)}I_{mt}^{1-\beta}z(m)+I_{mt}^{\alpha}f_{3,i}(t)$$

(9) ko‘rinishdagi 2-tur Volterra integral tenglamasiga ega bo‘lamiz. Soddalik maqsadida quyidagicha belgilash kiritaylik:

(10) $$f_{4,i}(t)=\frac{(t-m)^{\alpha-1}}{\Gamma(\alpha)}I_{mt}^{1-\alpha}z(m)+\frac{(t-m)^{\alpha-2}}{\Gamma(\alpha-1)}I_{mt}^{2-\alpha}z(m)-$$

$$-\tilde{\lambda}_i\frac{(t-m)^{\alpha-1}}{\Gamma(\alpha)}I_{mt}^{1-\beta}z(m)+I_{mt}^{\alpha}f_{3,i}(t).$$

U holda (9) tenglama ushbu ko‘rinishga keladi:

(11) $$z(t)-\frac{\tilde{\lambda}_i}{\Gamma(\alpha-\beta)}\int_m^t(t-\eta)^{\alpha-\beta-1}z(\eta)d\eta=f_{4,i}(t).$$

Oxirgi tenglama yechimini ketma-ket yaqinlashish usulidan foydalanib topamiz. Nolinchi yaqinlashish sifatida $f_{4,i}(t)$ni qabul qilamiz:

$$z_0(t)=f_{4,i}(t).$$

Birinchi va ikkinchi yaqinlashishlarni mos holda quyidagi formulalar orqali aniqlaymiz:

$$z_1(t)=f_{4,i}(t)+\tilde{\lambda}_i\int_m^t K(t,\eta)f_{4,i}(\eta)d\eta,$$

$$z_2(t)=f_{4,i}(t)+\tilde{\lambda}_i\int_m^t K(t,\tau)\left[f_{4,i}(\tau)+\tilde{\lambda}_i\int_m^\tau K(\tau,\eta)f_{4,i}(\eta)d\eta\right]d\tau=$$

$$=f_{4,i}(t)+\tilde{\lambda}_i\int_m^t K(t,\eta)f_{4,i}(\eta)d\eta+\tilde{\lambda}_i^2\int_m^t K_2(t,\eta)f_{4,i}(\eta)d\eta.$$

n-yaqinlashishni esa

$$z_n=f_{4,i}(t)+\tilde{\lambda}_i\int_m^t\sum_{j=0}^n K_j(t,\eta)f_{4,i}(\eta)d\eta,\quad j=1,2,\dots.$$

formuladan foydalanib topamiz, bu yerda $K_j(t,\eta)$–iteratsiyalangan yadrolar bo‘lib,

$$K_1(t,\eta)=K(t,\eta)=\frac{(t-\eta)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)},$$

$$K_j(t,\eta)=\int_\eta^t K(t,\tau)K_{j-1}(\tau,\eta)d\tau.$$

Dastlab $K_2(t,\eta)$ni hisoblaylik:

$$K_2(t,\eta)=\int_{\eta}^{t}K(t,\tau)K(\tau,\eta)d\tau=\int_{\eta}^{t}\frac{(t-\tau)^{\alpha-\beta-1}(\tau-\eta)^{\alpha-\beta-1}}{\Gamma^2(\alpha-\beta)}d\tau,\ j=2,3,\dots .$$

Ushbu almashtirishni bajaramiz: $\tau=(t-\eta)s+\eta$. U holda beta va gamma funksiya xossalaridan foydalansak [4],

$$K_2(t,\eta)=\int_0^1\frac{(t-\eta)^{2(\alpha-\beta)-1}(1-s)^{(\alpha-\beta)-1}s^{(\alpha-\beta)-1}}{\Gamma^2(\alpha-\beta)}ds=\frac{(t-\eta)^{2(\alpha-\beta)-1}}{\Gamma^2(\alpha-\beta)}B(\alpha-\beta,\alpha-\beta)=$$

$$=\frac{(t-\eta)^{2(\alpha-\beta)-1}\Gamma^2(\alpha-\beta)}{\Gamma^2(\alpha)\Gamma(2(\alpha-\beta))}=\frac{(t-\eta)^{2(\alpha-\beta)-1}}{\Gamma(2(\alpha-\beta))}.$$

Matematik induksiya usulidan foydalanib, ko'rsatish mumkinki, $K_j(t,\eta)$ uchun quyidagi ifoda o'rinli:

$$K_j(t,\eta)=\frac{(t-\eta)^{j(\alpha-\beta)-1}}{\Gamma(j(\alpha-\beta))},\quad j=1,2,3,\dots .$$

Endi $R(t,\eta;\tilde{\lambda}_i)$ rezolventani tuzamiz:

$$R(t,\eta;\tilde{\lambda}_i)=\sum_{j=1}^{\infty}\tilde{\lambda}_i^{j-1}K_j(t,\eta)=\sum_{j=1}^{\infty}\frac{\tilde{\lambda}_i^{j-1}(t-\eta)^{j(\alpha-\beta)-1}}{\Gamma(j(\alpha-\beta))}=\sum_{j=0}^{\infty}\frac{\tilde{\lambda}_i^{j}(t-\eta)^{j(\alpha-\beta)+(\alpha-\beta)-1}}{\Gamma(j(\alpha-\beta)+(\alpha-\beta))}.$$

$E_{\alpha,\beta}(z)$–Mittag-Leffler funksiyasining yoyilmasidan foydalansak [5], rezolventani quyidagicha yozishimiz mumkin:

$$R(t,\eta;\tilde{\lambda}_i)=(t-\eta)^{(\alpha-\beta)-1}E_{\alpha-\beta,\alpha-\beta}\left[\tilde{\lambda}_i(t-\eta)^{\alpha-\beta}\right].$$

U holda 2-tur Volterra integral tenglamalari nazariyasiga [6] asosan, (11) tenglama yechimi quyidagi formula orqali aniqlanadi:

$$z(t)=f_{4,i}(t)+\tilde{\lambda}_i\int_m^t(t-\eta)^{\alpha-\beta-1}E_{\alpha-\beta,\alpha-\beta}\left[\tilde{\lambda}_i(t-\eta)^{\alpha-\beta}\right]f_{4,i}(\eta)d\eta. \tag{12}$$

Endi (6) va (12) tengliklarga asosan quyidagi tenglamalar sistemasini tuzamiz:

$$\begin{cases}x(t)+\lambda_1y(t)=f_{4,1}(t)+(a+\lambda_1c)\int_m^t(t-\eta)^{\alpha-1}E_{\alpha-\beta,\alpha-\beta}\left[(a+\lambda_1c)(t-\eta)^{\alpha-\beta}\right]f_{4,1}(\eta)d\eta\\ x(t)+\lambda_2y(t)=f_{4,2}(t)+(a+\lambda_2c)\int_m^t(t-\eta)^{\alpha-1}E_{\alpha-\beta,\alpha-\beta}\left[(a+\lambda_2c)(t-\eta)^{\alpha-\beta}\right]f_{4,2}(\eta)d\eta\end{cases}.$$

Oxirgi tenglamalar sistemasini algebraik qo'shish usulidan $x(t)$ va $y(t)$ noma'lum funksiyalarni

$$x(t)=f_{4,1}(t)+(a+\lambda_1c)\int_m^t(t-\eta)^{\alpha-1}E_{\alpha-\beta,\alpha-\beta}\left[(a+\lambda_1c)(t-\eta)^{\alpha-\beta}\right]f_{4,1}(\eta)d\eta-\lambda_1y(t);$$

$$y(t)=\frac{1}{\lambda_1-\lambda_2}\Big\{f_{4,1}(t)-f_{4,2}(t)+(a+\lambda_1 c)\int_m^t (t-\eta)^{\alpha-\beta-1}E_{\alpha-\beta,\alpha-\beta}\left[(a+\lambda_1 c)(t-\eta)^{\alpha-\beta}\right]f_{4,1}(\eta)d\eta-$$

$$-(a+\lambda_2 c)\int_m^t (t-\eta)^{\alpha-\beta-1}E_{\alpha-\beta,\alpha-\beta}\left[(a+\lambda_2 c)(t-\eta)^{\alpha-\beta}\right]f_{4,2}(\eta)d\eta\Big\}$$

ko‘rinishida bir qiymatli aniqlanadi.

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