KASR TARTIBLI DIFFERENSIAL OPERATORNI O'Z ICHIGA OLUVCHI CHIZIQLI DIFFERENSIAL TENGLAMALAR SISTEMASINI YECHISHNING BIR USULI HAQIDA

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Annotatsiya. *Ushbu maqolada kasr tartibli differensial operatorni o'z ichiga oluvchi chiziqli differensial tenglamalar sistemasi uchun Koshi masalasi yechimi Dalamber usulidan foydalanib aniqlangan.*

Quyidagi

(1) $$\begin{cases} D_{mt}^{\alpha}x(t)=ax+by+f_1(t), \\ D_{mt}^{\alpha}y(t)=cx+dy+f_2(t), \end{cases} \qquad t>m$$

sistemaning

$$\lim_{t\to+m} I_{mt}^{1-\alpha}x(t)=x_0,\quad \lim_{t\to+m} I_{mt}^{1-\alpha}y(t)=y_0,$$

(2) $$\lim_{t\to+m} I_{mt}^{2-\alpha}x(t)=x_1,\quad \lim_{t\to+m} I_{mt}^{2-\alpha}y(t)=y_1$$

shartlarni qanoatlantiruvchi yechimini topish masalasini qaraylik, bu yerda D_{mt}^{β} va I_{mt}^{γ} kasr tartibli Riman-Liuvill ma'nosidagi differensial va integral operatorlar [1]:

$$D_{mt}^{\alpha}f(t)=\frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_m^t (t-z)^{1-\alpha} f(z)dz,$$

$$I_{mt}^{1-\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_m^t (t-z)^{-\alpha} f(z)dz,$$

$$I_{mt}^{2-\alpha}f(t)=\frac{1}{\Gamma(2-\alpha)}\int_m^t (t-z)^{1-\alpha} f(z)dz,$$

$\Gamma(g)$–Eylerning gamma funksiyasi [2], $\alpha,a,b,c,d,x_0,x_1y_0,y_1,$–berilgan haqiqiy sonlar bo'lib, $1<\alpha<2$; $f_1(t)$ va $f_2(t)$–berilgan funksiyalar, $x(t)$ va $y(t)$lar esa noma'lum funksiyalar.

$\{(1),(2)\}$ Koshi masalasining yechimini topish bilan shug'ullanamiz. Shu maqsadda (1) ning ikkinchi tenglamasini λ songa ko'paytirib so'ngra birinchi tenglamaga qo'shamiz. Natijada, yig'indini differensiallash formulasini e'tiborga olgan holda ushbu

(3) $$D_{mt}^{\alpha}\left[x(t)+\lambda y(t)\right]=(a+\lambda c)x+(b+\lambda d)y+f_1(t)+\lambda f_2(t)$$

tenglikni hosil qilamiz.

(3) tenglikning o'ng tomonidagi birinchi hadni qavsdan tashqariga chiqarib

(4) $$D_{mt}^{\alpha}\left[x(t)+\lambda y(t)\right]=(a+\lambda c)\left(x+\frac{b+\lambda d}{a+\lambda c}y\right)+f_1(t)+\lambda f_2(t)$$

tenglikni hosil qilamiz.

λ sonni shunday tanlaylikki, u ushbu $\dfrac{b+\lambda d}{a+\lambda c}=\lambda$ tenglamaning yechimi bo'lsin. Faraz qilaylik, λ_1 va λ_2 oxirgi tenglamaning ildizlari bo'lib, $\lambda_1 \neq \lambda_2$ bo'lsin. U holda bu sonlarni (4) ga qo'yib, ushbu

(5) $$D_{mt}^{\alpha}(x(t)+\lambda_i y(t))=(a+\lambda_i c)\left(x+\frac{b+\lambda_i d}{a+\lambda_i c}y\right)+f_1(t)+\lambda_i f_2(t),\ i=1,2$$

ko'rinishdagi $x(t)+\lambda_i y(t)$ noma'lumlarga nisbatan chiziqli ikkita tenglamaga ega bo'lamiz. Soddalik maqsadida quyidagi belgilashlarni kiritaylik:

(6) $$x(t)+\lambda y(t)=z(t),\quad f_1(t)+\lambda_i f_2(t)=f_{3,i}(t),\quad a+\lambda_i c=\tilde{\lambda}_i;\ i=1,2.$$

U holda (5) tenglama ushbu ko'rinishga keladi:

(7) $$D_{mt}^{\alpha}z(t)=\tilde{\lambda}_i z(t)+f_{3,i}(t),\qquad i=1,2.$$

Ushbu

(8) $$I_{mt}^{\alpha}D_{mt}^{\alpha}z(t)=z(t)-\frac{(t-m)^{\alpha-1}}{\Gamma(\alpha)}I_{mt}^{1-\alpha}z(m)-\frac{(t-m)^{\alpha-2}}{\Gamma(\alpha-1)}I_{mt}^{2-\alpha}z(m)$$

formulani e'tiborga olgan holda (7) tenglikning har ikki tomoniga I_{mt}^{α} operatorni tatbiq qilamiz. Natijada (2) shartlarni e'tiborga olsak, $z(t)$ noma'lum funksiyaga nisbatan

$$z(t)-\frac{\tilde{\lambda}_i}{\Gamma(\alpha)}\int_m^t (t-\eta)^{\alpha-1}z(\eta)d\eta=I_{mt}^{\alpha}f_{3,i}(t)+$$

(9) $$+\frac{(t-m)^{\alpha-1}}{\Gamma(\alpha)}I_{mt}^{1-\alpha}(x_0+\lambda_i y_0)+\frac{(t-m)^{\alpha-2}}{\Gamma(\alpha-1)}I_{mt}^{2-\alpha}(x_1+\lambda_i y_1)$$

(9) ko'rinishdagi 2-tur Volterra integral tenglamasiga ega bo'lamiz. Soddalik maqsadida quyidagicha belgilash kiritaylik:

(10) $$f_{4,i}(t)=I_{at}^{\alpha}f_{3,i}(t)+\frac{(t-m)^{\alpha-1}}{\Gamma(\alpha)}I_{mt}^{1-\alpha}(x_0+\lambda_i y_0)+\frac{(t-m)^{\alpha-2}}{\Gamma(\alpha-1)}I_{mt}^{2-\alpha}(x_1+\lambda_i y_1).$$

U holda (9) tenglama ushbu ko'rinishga keladi:

(11) $$z(t)-\frac{\tilde{\lambda}_i}{\Gamma(\alpha)}\int_m^t (t-\eta)^{\alpha-1}z(\eta)d\eta=f_{4,i}(t).$$

Oxirgi tenglama yechimini kema-ket yaqinlashish usulidan foydalanib topamiz. Nolinchi yaqinlashish sifatida $f_{4,i}(t)$ ni qabul qilamiz.

$$z_0(t)=f_{4,i}(t).$$

Birinchi va ikkinchi yaqinlashishlarni mos holda quyidagi formulalar orqali aniqlaymiz:

$$z_1(t)=f_{4,i}(t)+\lambda_i\int_m^t K(t,\eta)f_{4,i}(\eta)d\eta,$$

$$z_2(t)=f_{4,i}(t)+\lambda_i\int_m^t K(t,\tau)\left[f_{4,i}(\tau)+\lambda_i\int_m^\tau K(\tau,\eta)f_{4,i}(\eta)d\eta\right]d\tau=$$

$$=f_{4,i}(t)+\lambda_i\int_m^t K(t,\eta)f_{4,i}(\eta)d\eta+\lambda_i^2\int_m^t K_2(t,\eta)\,f_{4,i}(\eta)d\eta.$$

n–yaqinlashishni esa

$$z_n=f_{4,i}(t)+\lambda_i\int_m^t\sum_{j=0}^n K_j(t,\eta)\,f_{4,i}(\eta)d\eta$$

formuladan foydalanib topamiz, bu yerda $K_j(t,\eta)$-iteratsiyalangan yadrolar bo‘lib,

$$K_1(t,\eta)=K(t,\eta),\quad K_j(t,\eta)=\int_\eta^t K(t,\tau)K_{j-1}(\tau,\eta)d\tau.$$

Dastlab, $K_2(t,\eta)$ ni hisoblaylik:

$$K_2(t,\eta)=\int_\eta^t K(t,\tau)K(\tau,\eta)d\tau=\int_\eta^t\frac{(t-\tau)^{\alpha-1}(\tau-\eta)^{\alpha-1}}{\Gamma^2(\alpha)}d\tau.\quad j=2,3,\dots.$$

Ushbu almashtirishni bajaramiz: $\tau=(t-\eta)s+\eta$. U holda beta va gamma funksiya xossalaridan foydalansak [2],

$$K_2(t,\eta)=\int_0^1\frac{(t-\eta)^{2\alpha-1}(1-s)^{\alpha-1}s^{\alpha-1}}{\Gamma^2(\alpha)}ds=\frac{(t-\eta)^{2\alpha-1}}{\Gamma^2(\alpha)}\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}ds=$$

$$=\frac{(t-\eta)^{2\alpha-1}}{\Gamma^2(\alpha)}B(\alpha,\alpha)=\frac{(t-\eta)^{2\alpha-1}\Gamma^2(\alpha)}{\Gamma^2(\alpha)\Gamma(2\alpha)}=\frac{(t-\eta)^{2\alpha-1}}{\Gamma(2\alpha)}.$$

Matematik induksiya usulidan foydalanib, ko‘rsatish mumkinki, $K_j(t,\eta)$ uchun quyidagi ifoda o‘rinli:

$$K_j(t,\eta)=\frac{(t-\eta)^{j\alpha-1}}{\Gamma(j\alpha)}.\quad j=1,2,3,\dots.$$

Endi $R(t,\eta;\lambda_i)$ rezolventani tuzamiz:

$$R(t,\eta;\lambda_i)=\sum_{j=1}^\infty\lambda_i^{j-1}K_j(t,\eta)=\sum_{j=1}^\infty\frac{\lambda_i^{j-1}(t-\eta)^{j\alpha-1}}{\Gamma(j\alpha)}=\sum_{j=0}^\infty\frac{\lambda_i^{j}(t-\eta)^{j\alpha+\alpha-1}}{\Gamma(j\alpha+\alpha)}.$$

$E_{\alpha,\beta}(z)$–Mittag-Leffler funksiyasining yoyilmasidan foydalansak [3], rezolventani quyidagicha yozishimiz mumkin:

$$R\left(t,\eta;\tilde{\lambda}_i\right)=\left(t-\eta\right)^{\alpha-1}E_{\alpha,\alpha}\left[\tilde{\lambda}_i\left(t-\eta\right)^{\alpha}\right].$$

U holda 2-tur Volterra integral tenglamalari nazariyasiga [4] asosan (11) tenglama yechimi ushbu formula orqali aniqlanadi:

(12) $$z(t)=f_{4,i}(t)+\tilde{\lambda}_i\int_m^t (t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[\tilde{\lambda}_i(t-\eta)^{\alpha}\right]f_{4,i}(\eta)d\eta .$$

Endi (6) va (12) tengliklarga asosan quyidagi tenglamalar sistemasini tuzamiz:

$$\begin{cases} x(t)+\lambda_1 y(t)=f_{4,1}(t)+(a+\lambda_1 c)\int_m^t (t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[(a+\lambda_1 c)(t-\eta)^{\alpha}\right]f_{4,1}(\eta)d\eta \\ x(t)+\lambda_2 y(t)=f_{4,2}(t)+(a+\lambda_2 c)\int_m^t (t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[(a+\lambda_2 c)(t-\eta)^{\alpha}\right]f_{4,2}(\eta)d\eta \end{cases}.$$

Oxirgi tenglamar sistemasini algebraik qo'shish usulidan $x(t)$ va $y(t)$ noma'lum funksiyalarni

$$x(t)=f_{4,1}(t)+(a+\lambda_1 c)\int_m^t (t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[(a+\lambda_1 c)(t-\eta)^{\alpha}\right]f_{4,1}(\eta)d\eta-\lambda_1 y(t);$$

$$y(t)=\frac{1}{\lambda_1-\lambda_2}\Big\{f_{4,1}(t)-f_{4,2}(t)+(a+\lambda_1 c)\int_m^t (t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[(a+\lambda_1 c)(t-\eta)^{\alpha}\right]f_{4,1}(\eta)d\eta-$$

$$-(a+\lambda_2 c)\int_m^t (t-\eta)^{\alpha-1}E_{\alpha,\alpha}\left[(a+\lambda_2 c)(t-\eta)^{\alpha}\right]f_{4,2}(\eta)d\eta\Big\}$$

ko'rinishda bir qiymatli aniqlanadi.

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