TO'RTINCHI TARTIBLI XUSUSIY HOSILALI DIFFERENSIAL TENGLAMA UCHUN BOSHLANG'ICH SHARTDA YUQORI TARTIBLI HOSILA QATNASHGAN MASALA UCHUN TO'G'RI TO'RTBURCHAK SOHADA BIR ARALASH MASALA

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Annotatsiya: *Ushbu maqolada to'rtinchi tartibli xususiy hosilali differensial tenglama uchun boshlang'**ich shartda yuqori tartibli hosila qatnashgan** masala uchun to'g'ri to'rtburchak sohada bir aralash masala yechimining mavjudligi va yagonaligi Furye usuli yordamida tadqiq qilingan.*

Kalit so'zlar:*To'rtinchi tartibli xususiy hosilali tenglama, aralash masala, Furye usuli.*

To'rtinchi tartibli xususiy hosilali differensial tenglamalar uchun to'g'ri to'rtburchakli sohada ko'plab tadqiqotlar olib borilgan [1-7]. Lekin [1-6] ishlarda va boshqalarda boshlang'ich shartlarda yuqori tartibli hosilalardan foydalanilmagan. Ushbu ishda boshlang'ich shartda yuqori tartibli hosila berilgan masala yechimining yagonaligi o'rganilgan.

$\Omega=\{(x,t):0<x<l,0<t<T\}$ sohada

$$Lu\equiv u_t+a^2u_{xxxx}=0 \qquad (1)$$

tenglamani qaraylik.

***A* masala.** Ω sohada (1) tenglamani qanoatlantiruvchi va quyidagi shartlarni bajaruvchi

$$u_t^{(k)}(x,0)=\varphi(x),\quad 0\le x\le l, \qquad (2)$$

$$u(0,t)=0,\ u(l,t)=0,\ u_{xx}(0,t)=0,\ u_{xx}(l,t)=0,\quad 0\le t\le T, \qquad (3)$$

$u(x,t)\in C_{x,t}^{3,0}(\overline{\Omega})\cap C_{x,t}^{4,k}(\Omega)$ funksiya topilsin, bu yerda $k\in N$, $\varphi(x)$ - berilgan funksiya.

1-teorema. *Agar A-masalaning yechimi mavjud bo'lsa, u yagonadir.*

Isbot. Teskarisidan faraz qilaylik. *A* masalaning ikkita $u_1(x,t)$ va $u_2(x,t)$ yechimlarga ega bo'lsin. Bu yechimlarning ayirmasi bir jinsli (1) tenglamani va (2)-(3) larga mos bir jinsli boshlang'ich va chegaraviy shartlarni qanoatlantiradi. Bu ayirmani $u(x,t)$ bilan belgilaymiz, ya'ni

$$u(x,t)=u_1(x,t)-u_2(x,t). \qquad (4)$$

Ma'lumki,

$$X_n(x)=\sqrt{\frac{2}{l}}\sin\lambda_n x,\ \lambda_n=\frac{n\pi}{l},\quad n=1,2,\ldots. \qquad (5)$$

funksiya $L_2(0,l)$ da to'la ortonormallangan sistemani tashkil etadi [8].

Quyidagi

$$d_n(t)=\int_0^l u(x,t)X_n(x)dx,\ n=1,2,\dots \qquad (6)$$

funksiyani qaraymiz. (6) ni quyidagi ko'rinishda yozib olamiz:

$$d_{n,\varepsilon}(t)=\int_\varepsilon^{l-\varepsilon} u(x,t)X_n(x)dx,\ 0<\varepsilon<l, \qquad (7)$$

bunda $(\varepsilon,l-\varepsilon)\neq\varnothing$.

(7) tenglikni t bo'yicha bir marta differensiallab,

$$d'_{n,\varepsilon}(t)=\int_\varepsilon^{l-\varepsilon} u_t(x,t)X_n(x)dx$$

tenglikni hosil qilamiz. (1) tenglamani e'tiborga olib

$$d^n_{n,\varepsilon}(t)=-\int_\varepsilon^{l-\varepsilon} u_{xxxx}(x,t)X_n(x)dx \qquad (8)$$

tenglikni hosil qilamiz. (8) tenglikni to'rt marta bo'laklab integrallab, so'ng $\varepsilon\to 0$ da limitga o'tamiz:

$$\lim_{\varepsilon\to0} d'_{n,\varepsilon}(t)=-\lim_{\varepsilon\to0}\left[X_n(l-\varepsilon)u_{xxx}(l-\varepsilon,t)-X_n(\varepsilon)u_{xxx}(\varepsilon,t)\right]+$$
$$+\lim_{\varepsilon\to0}\left[X'_n(l-\varepsilon)u_{xx}(l-\varepsilon,t)-X'_n(\varepsilon)u_{xx}(\varepsilon,t)\right]-$$
$$-\lim_{\varepsilon\to0}\left[X''_n(l-\varepsilon)u_x(l-\varepsilon,t)-X''_n(\varepsilon)u_x(\varepsilon,t)\right]+$$
$$+\lim_{\varepsilon\to0}\left[X'''_n(l-\varepsilon)u(l-\varepsilon,t)-X'''_n(\varepsilon)u(\varepsilon,t)\right]-\lim_{\varepsilon\to0}\lambda_n^4\int_\varepsilon^{l-\varepsilon} X_n(x)u(x,t)dx.$$

Oxirgi tenglikdan (3) ga mos bir jinsli shartlar hamda $X_n(x)$ funksiyaning xossalariga asosan quyidagi natija kelib chiqadi:

$$d'_n(t)=-\lambda_n^4\int_0^l u(x,t)X_n(x)dx,\ n=1,2,\dots\ .$$

(4) ni e'tiborga olib quyidagi oddiy differensial tenglamaga kelamiz:

$$d'_n(t)+\lambda_n^4 d_n(t)=0,\quad n=1,2,\dots\ . \qquad (9)$$

(2) shartga mos bir jinsli shartdan (6) ga asosan

$$d_n^{(k)}(0)=0,\quad n=1,2,\dots \qquad (10)$$

boshlang'ich shartlarni hosil qilamiz.

Ma'lumki, (9) oddiy differensial tenglamaning umumiy yechimi quyidagi ko'rinishda aniqlanadi:

$d_n(t)=C_{1n}e^{-\lambda_n^4 t}.$

Bu yechimni (10) shartga bo`ysundirib $C_{1n}=0$ ga ega bo'lamiz, ya'ni barcha $\forall t\in[0,T]$ uchun $d_n(t)=0$. U holda (6) tenglik quyidagi ko'rinishni oladi:

$$\int_0^l u(x,t)X_n(x)dx=0,\quad n=0,1,\dots\ . \qquad (11)$$

Ma'lumki, (5) to'la ortonormal sistemani tashkil etadi. (11) dan esa $u(x,t)\equiv 0$ ekanligi kelib chiqadi. (4) ga asosan $u_1(x,t)\equiv u_2(x,t)$ tenglik kelib chiqadi. Bundan ko'rinadiki, agar A masalaning yechimi mavjud bo'lsa u yagonadir. 1-teorema isbotlandi.

A **masala yechimining mavjudligi.**

2-teorema. *Agar* $\varphi(x)\in C[0,1]$, $\varphi'(x)\in L_2(0,1)$ *bo'lib,* $\varphi(0)=0,$ $\varphi(l)=0$ *tengliklar bajarilsa, u holda* A *masalaning yechimi mavjud va u quyidagi ko'rinishda aniqlanadi:*

$$u(x,t)=\sqrt{\frac{2}{l}}\sum_{n=1}^{+\infty}a_n e^{-\lambda_n^4 a^2 t}\sin\frac{\pi n}{l}x,$$

bu yerda $a_n=\sqrt{\frac{2}{l}}\frac{1}{a^{2k}}\lambda_n^{-4k}\int_0^l\varphi(x)\sin\frac{\pi n}{l}dx.$

Isbot. A masalaning yechimini mavjudligini isbotlash uchun o'zgaruvchilarni ajraladigan usulini qo'llaymiz.

(1) tenglamaning yechimini quyidagi ko'rinishda izlaymiz:

$$u(x,t)=X(x)T(t). \qquad (12)$$

(12) ni (1) tenglamaga qo'yib, ushbu tenglikka ega bo'lamiz:

$$\frac{X^{(IV)}(x)}{X(x)}=-\frac{T'(t)}{a^2T(t)}.$$

Oxirgi tenglikda tenglikning chap tomoni faqat x ga, o'ng tomoni esa faqat t o'zgaruvchiga bog'liqdir. Bundan kelib chiqadiki, tenglik o'rinli bo'lishi uchun ikki nisbat ham biror o'zgarmas songa teng bo'lishi kerak. Bu o'zgarmasni λ^4 orqali belgilaymiz. Unda oxirgi tenglikdan, $X(x)$ noma'lum funksiyaga nisbatan quyidagi oddiy differensial tenglama kelib chiqadi:

$$X^{(IV)}(x)-\lambda^4X(x)=0. \qquad (13)$$

(3) chegaraviy shartlardan esa

$$X(0)=X(l)=X''(0)=X''(l)=0$$

chegaraviy shartlarga ega bo'lamiz. Tekshirib ko'rish qiyin emaski, (13) tenglamaning umumiy yechimi

$$X(x)=C_1e^{\lambda x}+C_2e^{-\lambda x}+C_3\cos\lambda x+C_4\sin\lambda x.$$

bo‘lib, uni chegaraviy shartlarga bo‘ysundirsak,

$$X_n(x)=C_4\sin\frac{n\pi}{l}x,\ \ n=0,1,\dots, \qquad (14)$$

natijani olamiz. (14) ni normallashtirib

$$X_n(x)=\sqrt{\frac{2}{l}}\sin\lambda_n x,\ \ \lambda_n=\frac{n\pi}{l},\ \ n=1,2,\dots$$

ifodani hosil qilamiz.

$T_n(t)$ ga nisbatan oddiy differensial tenglama quyidagi ko‘rinishga bo’ladi:

$$T_n'(t)+\lambda_n^4a^2T_n(t)=0.$$

Bu oddiy differensial tenglama umumiy yechimi

$$T_n(t)=a_ne^{-\lambda_n^4a^2t} \qquad (15)$$

formula orqali aniqlanadi, bu yerda $\lambda_n=\frac{n\pi}{l}$.

U holda *A* masalaning yechimi

$$u(x,t)=\sqrt{\frac{2}{l}}\sum_{n=1}^{+\infty}a_ne^{-\lambda_n^4a^2t}\sin\frac{\pi n}{l}x \qquad (16)$$

ko‘rinishni oladi, bu yerda

$$a_n=\sqrt{\frac{2}{l}}\frac{1}{a^{2k}}\lambda_n^{-4k}\int_0^l\varphi(x)\sin\frac{\pi n}{l}dx. \qquad (17)$$

1-lemma. *$\forall t\in[0,T]$ va $n=1,2,\dots$ lar uchun quyidagi baholashlar o‘rinli:*

$$|T_n(t)|\le|a_n|,\quad |T_n'(t)|\le C_1\lambda_n^4|a_n|,$$

$$\left|T_n^{(k)}(t)\right|\le C_2\lambda_n^{4k}|a_n|, \qquad (18)$$

bu yerda C_1, C_2 - musbat o‘zgarmas sonlar.

Isbot. (15) dan quyidagiga egamiz:

$$|T_n(t)|=\left|a_ne^{-\left(\frac{\pi n}{l}\right)^4a^2t}\right|=|a_n|\left|e^{-\left(\frac{\pi n}{l}\right)^4a^2t}\right|=|a_n|\left|\frac{1}{e^{\left(\frac{\pi n}{l}\right)^4a^2t}}\right|\le|a_n|.$$

(15) dan quyidagi

$$T_n'(t)=-\left(\frac{\pi n}{l}\right)^4a^2a_ne^{-\left(\frac{\pi n}{l}\right)^4a^2t}$$

ifodani topib, so‘ngra uni baholaymiz:

$$\left|T_n'(t)\right| = \left|-\left(\frac{\pi n}{l}\right)^4 a^2 a_n e^{-\left(\frac{\pi n}{l}\right)^4 a^2 t}\right| = \left|a_n\right|\left|e^{-\left(\frac{\pi n}{l}\right)^4 a^2 t}\right|\left|\left(\frac{\pi n}{l}\right)^4 a^2\right| =$$

$$= \left|a_n\right|\left|\frac{1}{e^{\left(\frac{\pi n}{l}\right)^4 a^2 t}}\right|\left|\left(\frac{\pi n}{l}\right)^4 a^2\right| \le \left|a_n\right|\left|\left(\frac{\pi n}{l}\right)^4 a^2\right| \le C_1 \lambda_n^4 \left|a_n\right|.$$

Endi (15) dan $T_n^{(k)}(t)$ ni quyidagi ko'rinishda topamiz:

$$T_n^{(k)}(t) = (-1)^k \left(\frac{\pi n}{l}\right)^{4k} a^2 a_n e^{-\left(\frac{\pi n}{l}\right)^4 a^2 t}.$$

$T_n^{(k)}(t)$ funksiyalar uchun quyidagilar o'rinli:

$$\left|T_n^{(k)}(t)\right| = \left|(-1)^k \left(\frac{\pi n}{l}\right)^{4k} a^2 a_n e^{-\left(\frac{\pi n}{l}\right)^4 a^2 t}\right| = \left|\left(\frac{\pi n}{l}\right)^{4k} a^2\right|\left|a_n\right|\left|e^{-\left(\frac{\pi n}{l}\right)^4 a^2 t}\right| \le C_2 \lambda_n^{4k} \left|a_n\right|.$$

1-lemma isbotlandi.

Endi topilgan (16) yechimni yaqinlashuvchi ekanligini va

$$\frac{\partial^4 u}{\partial x^4} = \sum_{n=1}^{\infty} \lambda_n^4 T_n(t) X_n(x), \tag{19}$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} T_n'(t) X_n(x), \tag{20}$$

$$\frac{\partial^k u}{\partial t^k} = \sum_{n=1}^{\infty} T_n^{(k)}(t) X_n(x) \tag{21}$$

qatorlarni yaqinlashuvchilikka tekshiramiz. Buning uchun (21) ni yaqinlashuvchiligini tekshirishyetarli bo'ladi. (21) qatorning yaqinlashishidan (19), (20) qatorning yaqinlashishi kelib chiqadi. Endi (21) qatorning yaqinlashuvchiligini ko'rib chiqamiz, u uchun

$$\sum_{n=1}^{\infty} \left|T_n^{(k)}(t)\right| \tag{22}$$

qator mojarant qator bo'ladi. 1-lemmadagi (18) ni (22) ga qo'yib va o'zgarmas C_2 ni qoldirib yozsak,

$$\sum_{n=1}^{\infty} \lambda_n^{4k} \left|a_n\right| \tag{23}$$

hosil qilamiz.

Endi (23) ga (17) tenglikni qo'yib, quyidagi qatorni hosil qilamiz:

$$\sqrt{\frac{2}{l}} \frac{1}{a^{2k}} \sum_{n=1}^{\infty} \left|\varphi_n\right|, \tag{24}$$

bu yerda $\varphi_n = \int_0^l \varphi(x)\sin\frac{\pi n}{l}dx$.

(24) qatorni yaqinlashuvchiligini ko'rsatish maqsadida $\varphi_n = \int_0^l \varphi(x)\sin\frac{\pi n}{l}xdx$ ni bir marta bo'laklab integrallaymiz:

$$\varphi_n = \int_0^l \varphi(x)\sin\frac{\pi n}{l}xdx = -\frac{l}{\pi n}\left[\left(\varphi(x)\cos\frac{\pi n}{l}x\right)\Big|_0^l - \int_0^l \varphi'(x)\cos\frac{\pi n}{l}xdx\right] =$$

$$= -\frac{l}{\pi n}\left(\varphi(l)\cos\pi n - \varphi(0)\cos 0\right) + \frac{l}{\pi n}\int_0^l \varphi'(x)\cos\frac{\pi n}{l}xdx =$$

$$= \frac{l}{\pi n}\int_0^l \varphi'(x)\cos\frac{\pi n}{l}xdx.$$

2-teorem shartlarini e'tiborga olib, oxirgi tenglikdan quyidagini hosil qilamiz:

$$\varphi_n = \frac{l}{\pi n}b_n, \qquad (25)$$

bu yerda $b_n = \int_0^l \varphi'(x)\cos\frac{\pi n}{l}xdx$.

(25) ni (24) ga qo'yib, o'zgarmaslarni tashlab quyidagi natijaga kelamiz

$$\sum_{n=1}^{\infty}\frac{|b_n|}{n}$$

Bu qatornig yaqinlashuvchanligi elementar tengsizlikdan kelib chiqadi, ya'ni

$$\frac{|b_n|}{n} \le \frac{1}{2}\left[b_n^2 + \frac{1}{n^2}\right],$$

va

$$\sum_{n=1}^{\infty} b_n^2, \quad \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Birinchi qatorning yaqinlashuvchiligi bo'lak-bo'lak uzluksiz bo'lgan $\varphi'(x)$ funksiya uchun Bessel tengsizligidan kelib chiqadi. Ikkinchi qator umumlashgan garmonik $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ qator bo'lib, u yaqinlashuvchidir. Demak, (21) qator tekis yaqinlashadi.

Shunday qilib biz (21) mojorant qatorning yaqinlashuvchiligini ko‘rsatdik. Demak, $\overline{\Omega}$ sohada (21) qatorlar absolyut va tekis yaqinlashadi. (21) ning yaqinlashuvchiligi (16), (19) va (20) qatorlarning yaqinlashuvchiligidan kelib chiqadi.

2-teoremaning shartlari bajarilsa, unda A masalaning $u(x,t)$ yechimini ikki marta t bo‘yicha va to‘rt marta x bo‘yicha hadlab differensiallanuvchi bo‘ladi, bundan kelib chiqadiki topilgan qatorlar $\overline{\Omega}$ sohasida absolyut va tekis yaqinlashadi. Buni esa $u(x,t)\in C_{x3}^{4,2}(\overline{\Omega})$ ko‘rinishda yozishimiz mumkin.

Teorema isbotlandi.

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