# YUKLANGAN ARALASH PARABOLIK TENGLAMA UCHUN INTEGRAL SHARTLI MASALA

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**Annotatsiya.** *Ushbu maqolada vaqt yo'nalishlari perpendikulyar bo'lgan aralash parabolik tenglama uchun integral shartli masalaning bir qiymatli yechilishi tadqiq qilingan.*

**Kalit so'zlar.** *Aralash parabolik tenglama, integral shart.*

$D$ orqali $y=0$ , $y=h$ va $x=-T$ to‘g‘ri chiziqlar bilan chegaralangan yarim polosani belgilaylik, bu yerda $h=const>0$, $T=const>0$. Bu sohada quyidagi yuklangan tenglamani qaraylik:

$$0=Lu\equiv\begin{cases} L_1u\equiv u_{xx}-u_y+\lambda\dfrac{d}{dy}u(0,y),(x,y)\in D_1=D\cap(x>0), \\ L_2u\equiv u_{yy}+u_x, \qquad (x,y)\in D_2=D\cap(x<0), \end{cases}$$

bu yerda $\lambda$ – berilgan haqiqiy son.

$Lu=0$ – $D$ sohada aralash parabolik tenglama bo‘lib, $D_1$ sohada to‘g‘ri parabolik, $D_2$ sohada esa teskari parabolikdir. Aralash parabolik tenglamalar uchun chegaraviy masalalar birinchi bo‘lib fransuz matematigi Mario Jevre [1] tomonidan o‘rganilgan. Keyinchalik, tadqiqotchilar tomonidan aralash parabolik tenglamalar uchun lokal va nolokal masalalar o‘rganishga bo‘lgan qiziqish ortdi. Jumladan, [2] ishda vaqt yo‘nalishlari almashinuvchi aralash parabolik tenglama uchun Jevre masalasi tadqiq qilingan bo‘lsa, [3] ishda aralash parabolik tenglama uchun turli lokal va nolokal shartli masalalar qo‘yilgan va o‘rganilgan. A.M.Naxushevning [4] ishi esa xarakteristik formalari o‘zgaruvchi ikkinchi tartibli parabolik tenglamalar uchun korrekt masalalar qo‘yish va masalalar korrektligi uchun zaruriy va yetarli shartlar aniqlashga bag‘ishlangan.

Dastlab, tadqiqotchilar tomonidan ikkinchi tartibli aralash parabolik tenglamalar qaralgan bo‘lsa, keyinchalik yuqori tartibli tenglamalar uchun masalalar tadqiqoti T.D.Djuraev, S.A.Tersenev, D.Amanov, S.V. Popov va ularning shogirdlari tomonidan rivojlantirildi.

So‘ngi vaqtlarda tadqiqotchilar tomonidan kasr tartibli differensial operatorlarni o‘z ichiga oluvchi aralash parabolik tenglamalar uchun ham tadqiqotlar olib borilmoqda. Jumladan, [5] ishda ikkinchi tartibli aralash parabolik tenglama uchun Jevre masalasi o‘rganilgan bo‘lsa, [6] ishda to‘rtinchi tartibli aralash parabolik tenglama uchun masalalar spektral analiz usuli bilan tadqiq qilingan.

Yuqoridagi ishlarda qaralayotgan tenglamalarning vaqt yo'nalishlari kollinear bo'lib, vaqt yo'nalishlari kollinear bo'lmagan aralash parabolik tenglamalar uchun masalalar kam o'rganilgan. [7] ishda vaqt yo'nalishlari perpendikulyar bo'lgan aralash parabolik tenglama uchun nolokal shartli masalalar tadqiq qilingan bo'lsa, [8],[9] ishlarda kasr tartibli differensial operatorlarni o'z ichiga oluvchi aralash parabolik tenglamalar uchun turli nolokal shartli masalalar o'rganilgan.

Mazkur maqolada vaqt yo'nalishlari perpendikulyar bo'lgan yuklangan aralash parabolik tenglama uchun bir integral shartli masalaning bir qiymatli yechilishi tadqiq qilinadi.

$I$ **masala.** $D$ sohaning yopig'ida aniqlangan, uzluksiz va chegaralangan shunday $u(x,y)$ funksiya topilsinki, u $D_1$ va $D_2$ sohalarda mos ravishda $L_1u=0$ va $L_2u=0$ tenglamalarni hamda quyidagi shartlarni qanoatlantirsin:

$$\lim_{x\to -0} u_x(x,y) = \lim_{x\to +0} u_x(x,y), \qquad 0<y<h;$$

$$u(x,0)=\varphi_1(x), \qquad 0\le x<+\infty; \qquad (1)$$

$$u(x,0)=\varphi_2(x), \qquad -T\le x\le 0; \qquad (2)$$

$$u(x,h)=a(x)\int_0^h u(x,y)dy+\varphi_3(x), \qquad -T\le x\le 0, \qquad (3)$$

bu yerda $a(x)$ va $\varphi_j(x),\ j=\overline{1,3}$ – berilgan uzluksiz funksiyalar bo'lib, $\varphi_1(0)=\varphi_2(0)$.

Qo'yilgan masalaning yechimi mavjud va yagonaligini isbotlaymiz. $I$ masalanng $u(x,y)$ yechimi mavjud bo'lsin deb faraz qilaylik Masala shartlariga asoslanib,

$$u(-0,y)=u(+0,y)=\tau(y), \qquad 0\le y\le h; \qquad (4)$$

$$\lim_{x\to -0} u_x(x,y) = \lim_{x\to +0} u_x(x,y)=\nu(y), \qquad 0<y<h \qquad (5)$$

belgilashlarni kiritaylik.

Ma'lumki, $L_1u=0$ tenglamaning $D_1$ sohaning yopig'ida aniqlangan, uzluksiz, chegaralangan hamda (1) va $\lim_{x\to +0} u_x(x,y)=\nu(y)$, $0\le y\le h$ shartlarni qanoatlantiruvchi yechimi quyidagi ko'rinishda aniqlanadi [7]:

$$u(x,y)=\int_0^{+\infty}\varphi_1(\xi)G(x,\xi;y)d\xi-\int_0^y \nu(\eta)G(x,0,y-\eta)d\eta+$$

$$+\int_0^y\int_0^{+\infty}\tau'(\eta)G(x,\xi,y-\eta)d\xi d\eta, \qquad (6)$$

bu yerda

$$G(x,\xi,y)=\frac{1}{2\sqrt{\pi y}}\left\{\exp\left[-\frac{(x-\xi)^2}{4y}\right]+\exp\left[-\frac{(x+\xi)^2}{4y}\right]\right\}.$$

(6) formulada $x\to+0$ da limitga o'tamiz. U holda

$$\int_0^{+\infty}G(0;\xi,y-\eta)d\xi=1$$

tenglikni e'tiborga olgan holda quyidagi tenglikka ega bo'lamiz:

$$\tau(y)=-\frac{1}{\sqrt{\pi}}\int_0^y\nu(\eta)(y-\eta)^{-1/2}d\eta+\int_0^{+\infty}G(0;\xi,y)\varphi_1(\xi)d\xi+\lambda\int_0^y\tau'(\eta)d\eta. \quad (7)$$

(7) tenglikni kasr tartibli integral operator ko'rinishidan [10] foydalanib va $\tau(0)=\varphi_1(0)$ ekanligini e'tiborga olib, quyidagicha yozib olamiz:

$$(1-\lambda)\tau(y)=-D_{0y}^{-1/2}\nu(y)+F(y), \quad (8)$$

bu yerda

$$F(y)=\int_0^{+\infty}G(0,\xi,y)\varphi_1(\xi)d\xi-\lambda\varphi_1(0).$$

(8) tenglikning har ikki tomoniga $D_{0y}^{1/2}$ differensial operatorni tatbiq qilib va $D_{0y}^{1/2}D_{0y}^{-1/2}g(y)=g(y)$ formulani e'tiborga olib [10], ushbu tenglikka ega bo'lamiz:

$$\nu(y)=-(1-\lambda)D_{0y}^{1/2}\tau(y)+D_{0y}^{1/2}F(y). \quad (9)$$

(9) – noma'lum $\tau(y)$ va $\nu(y)$ funksiyalar orasidagi $D_1$ sohadan olingan funksional munosabatdir.

Endi masala shartlarini va (4),(5) belgilashlarni hisobga olib, $L_2u=0$ tenglama va (2), (3) shartlarda $x$ ni nolga intiltiramiz. Natijada

$$\tau''(y)+\nu(y)=0,\quad 0<y<h \quad (10)$$

tenglama va quyidagi shartlarga ega bo'lamiz:

$$\tau(0)=\varphi_1(0),\qquad \tau(h)=a(0)\int_0^h\tau(y)dy+\varphi_3(0). \quad (11)$$

(9) tenglikni e'tiborga olsak, (10) tenglikdan $\tau(y)$ noma'lum funksiyaga nisbatan

$$\tau''(y)=\frac{1-\lambda}{\sqrt{\pi}}\frac{d}{dy}\int_0^y(y-t)^{-1/2}\tau(t)dt+F_1(y) \quad (12)$$

ko'rinishdagi integro-differensial tenglamaga ega bo'lamiz, bu yerda

$$F_1(y)=\frac{1}{\sqrt{\pi}}\frac{d}{dy}\int_0^y(y-t)^{-1/2}F(t)dt.$$

Shunday qilib qo'yilgan masala (12) tenglamaning (11) shartlarni qanoatlantiruvchi yechimini topish masalasiga ekvivalent (yechimga ega bo'lish ma'nosida) keltirildi. Agar {(12),(11)} masaladan $\tau(y)$ ni bir qiymatli aniqlasak, u holda $\nu(y)$ funksiya (9) tenglik bilan aniqlanadi. Shundan so'ng masalaning yechimi $D_1$ sohada (6) formula bilan $D_2$ sohada esa $L_2 u=0$ tenglama uchun birinchi chegaraviy masalaning yechimi sifatida aniqlanadi.

Shuning uchun bundan buyon {(12),(11)} masalaning bir qiymatli yechilishi masalasi bilan shug`ullanamiz.

Bu yerda quyidagi ikki holni qaraymiz:

1-hol. $\tau(y)$ $\lambda=1$;

2-hol. $\lambda\neq 1$.

Dastlab, 1-holni qaraylik. Bu holda (12) tenglama

$$\tau''(y)=F_1(y) \tag{13}$$

ko`rinishda bo'lib, uni ketma-ket ikki marta integrallab va $\tau(0)=\varphi_1(0)$ ekanini e'tiborga olib, $\tau(y)$ funksiyani quyidagi ko'rinishda topamiz:

$$\tau(y)=\int_0^y (y-t)F_1(t)dt+Cy+\varphi_1(0)\,, \tag{14}$$

bu yerda $C$ – ixtiyoriy noma'lum son.

$C$ noma`lumni topish maqsadida (14) formula bo'yicha $\tau(h)$ va $\int_0^h \tau(y)dy$ larni hisoblaymiz:

$$\tau(h)=\int_0^h (h-t)F_1(t)dt+hC+\varphi_1(0),$$

$$\int_0^h \tau(y)dy=\frac{1}{2}\int_0^h (h-t)^2 F_1(t)dt+\frac{h^2C}{2}+\varphi_1(0)h.$$

Bularni (11) shartlarning ikkinchisiga qo'yib, quyidagi tenglikka kelamiz:

$$C\left[a(0)\frac{h}{2}-1\right]h=\int_0^h (h-t)F_1(t)dt-\frac{a(0)}{2}\int_0^h (h-t)^2 F_1(t)dt+$$

$$+\left[1-a(0)h\right]\varphi_1(0)-\varphi_3(0). \tag{15}$$

Agar $a(0)$ va $h$ sonlar uchun quyidagi tengsizlik

$$a(0)\neq\frac{2}{h} \tag{16}$$

bajarilgan bo‘lsa, (15) tenglikdan $C$ noma’lum son bir qiymatli topiladi. $C$ noma’lumning topilgan qiymatini (14) tenglikka qo’ysak, $\tau(y)$ funksiya ma’lum bo’ladi. $\tau(y)$ ning topilgan qiymatini (9) tenglikka qo’yib, $\nu(y)$ funksiyani aniqlaymiz.

Endi 2-holni qaraylik. (12) tenglamaning umumiy yechimini topish maqsadida undagi $y$ ni $z$ bilan almashtiramiz. So‘ngra hosil bo‘lgan tenglikni $z$ bo‘yicha $[0,y]$ oraliqda ikki marta integrallaymiz. Natijada, $\tau(0)=\varphi_1(0)$ ekanini e’tiborga olib, $\tau(y)$ funksiyaga nisbatan quyidagi integral tenglamaga ega bo‘lamiz,

$$\tau(y)-\frac{2(1-\lambda)}{\sqrt{\pi}}\int_0^y \tau(y)(y-t)^{1/2}dt=F_2(y)+Cy,$$

bu yerda $C$ – ixtiyoriy o‘zgarmas son $F_2(y)=\int_0^y (y-t)F_1(t)dt+\varphi_1(0)$.

Ma’lumki, bu integral tenglamaning yechimi

$$\tau(y)=\frac{d}{dy}\int_0^y E_{3/2,1}\left[(1-\lambda)(y-t)^{3/2}\right]\left[F_2(t)+Ct\right]dt \qquad (17)$$

formula bilan aniqlanadi [11], bu yerda $E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$ – Mittag-Leffler funksiyasi, $\Gamma(z)$ esa Eylerning gamma funksiyasi [10].

(17) tenglikda differensiallash amalini bajarib va

$$E_{3/2,1}(0)=1,\quad \frac{\partial}{\partial y}E_{3/2,1}\left[(1-\lambda)(y-t)^{3/2}\right]=-\frac{\partial}{\partial t}E_{3/2,1}\left[(1-\lambda)(y-t)^{3/2}\right],$$

$$\int_0^y E_{3/2,1}\left[(1-\lambda)(y-t)^{3/2}\right]dt=y\,E_{3/2,2}\left[(1-\lambda)y^{3/2}\right],\quad F_2(0)=0$$

tengliklarni e’tiborga olib, $\tau(y)$ni quyidagi ko‘rinishda topamiz:

$$\tau(y)=CyE_{3/2,2}\left[(1-\lambda)y^{3/2}\right]+\int_0^y E_{3/2,1}\left[(1-\lambda)(y-t)^{3/2}\right]F_2'(t)dt. \qquad (18)$$

(18) formula bo‘yicha $\tau(h)$ va $\int_0^h \tau(y)dy$ larni hisoblaymiz:

$$\tau(h)=C\,hE_{3/2,2}\left[(1-\lambda)h^{3/2}\right]+\int_0^h E_{3/2,1}\left[(1-\lambda)(h-t)^{3/2}\right]F_2'(t)dt,$$

$$\int_0^h \tau(y)dy=Ch^2E_{3/2,3}\left[(1-\lambda)h^{3/2}\right]+\int_0^h E_{3/2,2}\left[(1-\lambda)(h-t)^{3/2}\right]F_2'(t)dt.$$

Bularni (11) shartlarning ikkinchisiga qo‘yib, quyidagi tenglikka kelamiz:

$$C\left\{E_{3/2,2}\left[(1-\lambda)h^{3/2}\right]-a(0)hE_{3/2,3}\left[(1-\lambda)h^{3/2}\right]\right\}h=$$

$$=a(0)\int_0^h E_{3/2,2}\left[(1-\lambda)(h-t)^{3/2}\right]F_2'(t)dt-\int_0^h E_{3/2,1}\left[(1-\lambda)(h-t)^{3/2}\right]F_2'(t)dt+\varphi_3(0).$$

Agar $a(0)$ va $h$ sonlar uchun quyidagi tengsizlik

$$E_{3/2,2}\left[(1-\lambda)h^{3/2}\right]-a(0)hE_{3/2,3}\left[(1-\lambda)h^{3/2}\right]\neq 0 \qquad (19)$$

bajarilgan bo‘lsa, (18) tenglikdan $C$ noma’lum son bir qiymatli topiladi.

1-**izoh.** Masalan, $a(0)h\le 2$ bo‘lganda (19) tengsizlik bajariladi.

$C$ noma’lumning (18) tenglikdan topilgan qiymatini (17) ga qo‘yib, {(12),(11)} masalaning yechimi bo‘lgan $\tau(y)$ funksiyani to‘lig‘icha aniqlaymiz.

$\tau(y)$ funksiya topilgandan so‘ng $\nu(y)$ funksiya (9) tenglik bilan aniqlanadi. Shundan so‘ng $I$ masalaning yechimi $D_1$ sohada (6) formula bilan topiladi, $D_2$ sohada esa $L_2u=0$ tenglamaning (2),(3) va $u(0,y)=\tau(y),0\le y\le h$ shartlarni qanoatlantiruvchi yechimi sifatida topiladi. Oxirgi masalani $I_0$ deb belgilaymiz va bir qiymatli echilishini isbotlaymiz.

Faraz qilaylik, $u(x,y)-I_0$ masalaning yechimi bo‘lsin. $u(x,h)=\varphi(x),-T\le x\le 0$ belgilash kiritaylik. U holda, $u(x,y)$ funksiyani $D_2$ sohada $L_2u=0$ tenglama uchun birinchi chegaraviy masalaning yechimi sifatida

$$u(x,y)=\int_0^h \tau(\eta)G(0,\eta;x,y)d\eta+$$

$$+\int_x^0 \varphi_2(\xi)G_\eta(\xi,0;x,y)d\xi-\int_x^0 \varphi(\xi)G_\eta(\xi,h;x,y)d\xi \qquad (20)$$

ko‘rinishda yozish mumkin bo‘ladi [13] , bu yerda

$$G(\xi,\eta;x,y)=\frac{1}{2\sqrt{\pi(\xi-x)}}\times$$

$$\times\sum_{n=-\infty}^{\infty}\left\{\exp\left[-\frac{(y-\eta-2n)^2}{4(\xi-x)}\right]-\exp\left[-\frac{(y+\eta-2n)^2}{4(\xi-x)}\right]\right\},\xi>x..$$

$u(x,h)=\varphi(x)$ belgilashni va (20) formulani e’tiborga olib, (3) shartdan

$$\varphi(x)+\int_x^0 \varphi(\xi)\left[a(x)K(x,\xi)\right]d\xi=f_1(x), \qquad -T\le x\le 0 \qquad (21)$$

integral tenglamaga ega bo‘lamiz, bu yerda

$$K(x,\xi)=-\frac{1}{2\sqrt{\pi(\xi-x)}}\sum_{n=-\infty}^{+\infty}\left[e^{\frac{-n^2}{\xi-x}}-e^{\frac{-(2n+h)^2}{4(\xi-x)}}+e^{\frac{-(n-h)^2}{\xi-x}}-e^{\frac{-(2n-h)^2}{4(\xi-x)}}\right],\ \xi>x$$

$$f_1(x)=a(x)\int_0^h\left[\int_0^1\tau(\eta)G(0,\eta;x,y)d\eta+\int_x^0\varphi_2(\xi)G(\xi,0;x,y)d\xi\right]dy+\varphi_3(x).$$

(21) – Volterrning ikkinchi tur integral tenglamasidir. Uning yadrosi $(1/2)$ tartibli sust maxsuslikka ega bo‘lib, $x\to\xi$ da o‘zini $(x-\xi)^{-1/2}$ funksiya kabi tutadi, o‘ng tomoni esa $C[-T,0]$ sinfga tegishli, ya’ni

$$a(x)K(x,\xi)=O(1)(\xi-x)^{-1/2},\qquad f_1(x)\in C[-T,0].$$

Shuning uchun (21) integral tenglama $[-T,0]$ oraliqda uzluksiz bo‘lgan yagona yechimga ega. Demak, $I_0$ masala ham yagona yechimga ega.

Shunday qilib, quyidagi teorema o‘rinli ekanligi isbotlandi.

**Teorema.** Agar $a(x)\neq 0,\ x\in[-T,0]$ bo‘lib, $a(0)$ va $h$ sonlar (19) tengsizlikni qanoatlantirsa, $I$ masala yagona yechimga ega bo‘ladi.

2-**izoh.** $a(x)\equiv 0,\ x\in[-T,0]$ bo‘lganda ham $I$ masalaning yechimi $D_2$ sohada (20) formula bilan aniqlanadi, faqat bunda $\varphi(x)=\varphi_3(x)$ deb olinadi.

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