KVARTERNION FUNKSIYALAR

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Kvarternionlarning topilishi matematikada eng yaxshi hujjatlashtirilgan kashfiyotlardan biridir. Umuman olganda, yirik matematik kashfiyotning sanasi va joylashuvi juda kam uchraydigan ma'lumot. Kvarternionlarga kelsak, ular irland matematigi U.R. Gamilton tomonidan 1843- yil 16- oktyabr kashf etilgan,bu ma`lumot Dublindagi Brum ko`pirigida o`yib yozilgan toshlavxada aks etgan. Aniq aytilishiga sabab o`zining o`g`liga yozgan maktubida bu ma`lumot keltirilgan. Gamilton kvarternionlarni ochishdan oldin kompleks sonlar nazariyasi bilan shug`ullangan. Kompleks sonlar maydonini kengaytirishga harakat qilgan.

Misol uchun, C R ustidagi maydon kengaytmasi. Haqiqatan ham, R C ning kichik to'plamidir, chunki R ning har bir elementi $a+ib$ ko'rinishida yozilishi mumkin, bu erda $b=0$. Nihoyat, qo'shimcha va R ning multiplikativ operatorlari $b=0$ bo'lganda C operatorlari bilan mos keladi. Umuman olganda, biz C ni R ustidagi o'lchovli vektor fazosi deb ataymiz, kompleks sonlarning maydoni 2 darajali maydon kengaytmasidir.Kvarternionlarning maxsus holatida H ikkita yangi j va k elementlarni qo'shish orqali tuziladi. Bu mavxum elementlar orasida quyidagi tenglik o`rinli:

$$i^2 = j^2 = k^2 = ijk = -1$$

Shunday qilib, Kvarternionlarning maydonni quyidagicha yozish mumkin. $H = \{ q = q_0 + q_1 i + q_2 j + q_3 k \mid q_t \in R, \quad i^2 = j^2 = k^2 = -1 \}$

Kvarternion sonni $q = q_0 + q_1 i + q_2 j + q_3 k$ ko`rinishda bo`lib ikki qismdan iborat skalyar qism $scal q = q_0$ va vektor qism $vek\, q = q_1 i + q_2 j + q_3 k$. Kvarternion sonni $q = q_0 + i|q - q_0|$ kabi ham belgilanadi.

1. Normasi :

$$\|q\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \qquad (1)$$

2. Moduli:

$$|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \qquad (2)$$

3. Vektor qismini moduli:

$$\langle q \rangle = |q - q_0| = \sqrt{q_1^2 + q_2^2 + q_3^2} \qquad (3)$$

4. Argumenti :

$$\arg q = \arccos\frac{q_0}{|q|} \quad (4)$$

5. Kvarternion sonning teskarisi:

$$q^{-1} = \frac{\overline{q}}{|q|^2} \quad (5)$$

Kvarternion sonni qo`shmasi vektor qismining ishorasi bilan farq qiladi va quyidagicha yoziladi:

$$\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$$

Kvarternionlarni qo`shish va ayrish kompleks sonlar bilan bir xil bajariladi ko`paytirish va bo`lish ozgina farqliroq bajariladi.[1][3]

Misol uchun

$$q = 2 + 3i + 4j + 5k \in H \quad (6)$$

$$|q| = \sqrt{2^2 + 3^2 + 4^2 + 5^2} = \sqrt{54} \quad (7)$$

$$q_0 = 2 \quad (8)$$

$$q - q_0 = 3i + 4j + 5k \quad (9)$$

$$\langle q \rangle = |q - q_0| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \quad (10)$$

$$\begin{aligned} \lambda &= q_0 + i|q - q_0| = 2 + i\sqrt{50} \\ \overline{\lambda} &= q_0 - i|q - q_0| = 2 - i\sqrt{50} \end{aligned} \quad (11)$$

Kvarternion argumentli ba`zi funksiyalarni ko`rib chiqamiz buning uchun karternionlar uchun Koshi integral formulasi[2] 2-ko`rinishidan

$$f(q) = \mathrm{Re}(f(\lambda)) + \frac{q - q_0}{|q - q_0|}\mathrm{Im}(f(\lambda)) \quad (12)$$

formulasidan foydalanamiz :

1. e^q kvarternion exponentasi

$$e^{q_0 + iq_1 + jq_2 + kq_3} = \mathrm{Re}(e^{q_0 + i|q - q_0|}) + \frac{q - q_0}{|q - q_0|}\mathrm{Im}(e^{q_0 + i|q - q_0|}) \quad (13)$$

yoki

$$e^{q_0 + iq_1 + jq_2 + kq_3} = e^{q_0}\left(\cos(|q - q_0|) + \frac{q - q_0}{|q - q_0|}\sin(|q - q_0|)\right) \quad (14)$$

(6)-(10) sonlar uchun (13) olamiz

$$e^{2 + i3 + j4 + k5} = e^2\left(\cos(\sqrt{50}) + \frac{i3 + j4 + k5}{\sqrt{54}}\sin(\sqrt{50})\right) \quad (15)$$

2. $ln(q)$ kvarternionning natural logarifmi

$$\ln(q)=\ln(q_0+q_1i+q_2j+q_3k)=\mathrm{Re}(\ln(q_0+i|q-q_0|))+\frac{q-q_0}{|q-q_0|}\mathrm{Im}(\ln(q_0+i|q-q_0|))$$

(16) yoki

$$\ln(q)=\ln(q_0+q_1i+q_2j+q_3k)=\ln(|q|)+\frac{q-q_0}{|q-q_0|}\arg(q_0+i|q-q_0|)\quad(17)$$

$$\arg(\lambda)=\arg(q_0+i|q-q_0|)=\begin{cases} arctg\left(\frac{q-q_0}{|q-q_0|}\right), q_0>0 \\ \pi+arctg\left(\frac{q-q_0}{|q-q_0|}\right), q_0<0\end{cases}\quad(18)$$

(6)-(11)sonlarni (18) ga qo`ysak

$$\arg(\lambda)=\arg(q_0+i|q-q_0|)=\arg\left(\frac{|q-q_0|}{q_0}\right)=arctg(\sqrt{50})\quad(19)$$

(16) va (18) ga ko`ra

$$\ln(2+3i+4j+5k)=\ln(\sqrt{54})+\frac{3i+4j+5k}{\sqrt{50}}arct(\sqrt{50})\quad(20)$$

3. $\sqrt{q}$ kvarternionning kvadrat ildizi

$$\sqrt{q_0+q_1i+q_2j+q_3k}=\mathrm{Re}\left(\sqrt{q_0+i|q-q_0|}\right)+\frac{q-q_0}{|q-q_0|}\mathrm{Im}\left(\sqrt{q_0+i|q-q_0|}\right)\quad(21)$$

yoki

$$\sqrt{q}=\sqrt{q_0+q_1i+q_2j+q_3k}=\pm\left(\sqrt{\frac{|q|+q_0}{2}}+\frac{q-q_0}{|q-q_0|}\sqrt{\frac{|q|-q_0}{2}}\right)\quad(22)$$

(6)-(10) sonlarga ko`ra hisoblasak

$$\sqrt{q}=\pm\left(\sqrt{\frac{\sqrt{54}+2}{2}}+\frac{3i+4j+5k}{\sqrt{50}}\sqrt{\frac{\sqrt{54}-2}{2}}\right)\quad(23)$$

4. q^2 kvarternionning kvadrati

(11*) formuladan foydalanamiz

$$(q_0+q_1i+q_2j+q_3k)^2=\mathrm{Re}((q_0+i|q-q_0|)^2)+\frac{q-q_0}{|q-q_0|}\mathrm{Im}((q_0+i|q-q_0|)^2)\quad(24)$$

yoki

$$q^2=(q_0+q_1i+q_2j+q_3k)^2=q_0^2-|q-q_0|^2+\frac{q-q_0}{|q-q_0|}2q_0|q-q_0|\quad(25)$$

Amallarni bajarsak

$$q^2=(q_0+q_1i+q_2j+q_3k)^2=2qq_0-q_0^2-|q-q_0|^2$$

(26) yoki

$$q^2=(q_0+q_1i+q_2j+q_3k)^2=2q_0q-|q|^2$$

(27)

5. $(a_0+a_1i+a_2j+a_3k)^{(b_0+b_1i-b_2j-b_3k)}$ **kvarternionning kvarternion darajasi**

$$(a_0+a_1i+a_2j+a_3k)^{(b_0+b_1i-b_2j-b_3k)}=\beta_0+\beta_1 i+\beta_2 j+\beta_3 k \qquad (28)$$

bu yerda

$$e^{\mu_0}\{(\cos\mu_1\cos\mu_2\cos\mu_3-\sin\mu_1\sin\mu_2\sin\mu_3)=\beta_0 \quad (\cos\mu_1\sin\mu_2\sin\mu_3+\sin\mu_1\cos\mu_2\cos\mu_3)e^{\mu_0}=\beta_1$$

$$e^{\mu_0}(\cos\mu_1\sin\mu_2\cos\mu_3-\sin\mu_1\cos\mu_2\sin\mu_3)=\beta_3 \quad (\cos\mu_1\sin\mu_2\sin\mu_3+\sin\mu_1\cos\mu_2\cos\mu_3)e^{\mu_0}=\beta_3$$

$$\mu_0=b_0\delta_0-b_1\delta_1+b_2\delta_2+b_3\delta_3\ , \qquad \mu_1=b_0\delta_1+b_1\delta_0-b_2\delta_3+b_3\delta_2$$

$$\mu_2=b_0\delta_1-b_1\delta_3-b_2\delta_0-b_3\delta_1\ , \qquad \mu_3=b_0\delta_3+b_1\delta_2-b_2\delta_1+b_3\delta_0$$

$$\delta=a_0+a_1i+a_2j+a_3k, \qquad \delta_0=\ln|\delta|=\frac{1}{2}\ln(\sqrt{a_0+a_1i+a_2j+a_3k})$$

$$\delta_1=\frac{a_1}{|a_1i+a_2j+a_3k|}\arccos\frac{a_0}{|a_0+a_1i+a_2j+a_3k|}$$

$$\delta_2=\frac{a_2}{|a_1i+a_2j+a_3k|}\arccos\frac{a_0}{|a_0+a_1i+a_2j+a_3k|}$$

$$\delta_3=\frac{a_3}{|a_1i+a_2j+a_3k|}\arccos\frac{a_0}{|a_0+a_1i+a_2j+a_3k|}$$

6. $\sin(q)$ **sinus kvaternion**

$$\sin(q_0+q_1i+q_2j+q_3k)=\mathrm{Re}(\sin(q_0+i|q-q_0|))+\frac{q-q_0}{|q-q_0|}\mathrm{Im}(\sin(q_0+i|q-q_0|))$$

(29)

yoki

$$\sin(q)=\sin(q_0+q_1i+q_2j+q_3k)=\sin(q_0)ch(|q-q_0|)+\frac{q-q_0}{|q-q_0|}\cos(q_0)sh(|q-q_0|)$$

(30)

(6)-(10) sonlarga ko`ra

$$\sin(2+3i+4j+5k)=\sin(2)ch(\sqrt{50})+\frac{3i+4j+5k}{\sqrt{50}}\cos(2)sh(\sqrt{50})$$

(31)

7. $cos(q)$ **kosinus kvaternion**

$$\cos\left(q_0+q_1i+q_2j+q_3k\right)=\mathrm{Re}(\cos(q_0+i\left|q-q_0\right|))+\frac{q-q_0}{\left|q-q_0\right|}\mathrm{Im}(\cos(q_0+i\left|q-q_0\right|))$$

(32)

yoki

$$\cos\left(q\right)=\cos\left(q_0+q_1i+q_2j+q_3k\right)=\cos(q_0)ch(\left|q-q_0\right|)+\frac{q-q_0}{\left|q-q_0\right|}\sin(q_0)sh(\left|q-q_0\right|)$$

(33)

(6)-(10) sonlarga ko`ra

$$\cos\left(2+3i+4j+5k\right)=\cos(2)ch(\sqrt{50})+\frac{3i+4j+5k}{\sqrt{50}}\sin(2)sh(\sqrt{50})$$

8. $tg\left(q\right)$ **tangens kvaternion**

$$tg\left(q_0+q_1i+q_2j+q_3k\right)=\mathrm{Re}(tg(q_0+i\left|q-q_0\right|))+\frac{q-q_0}{\left|q-q_0\right|}\mathrm{Im}(tg(q_0+i\left|q-q_0\right|))$$

(34)

yoki

$$tg\left(q\right)=\frac{1}{\cos^2\left(q_0\right)+sh^2(\left|q-q_0\right|)}\left(\sin(q_0)\cos(q_0)+\frac{q-q_0}{\left|q-q_0\right|}sh(\left|q-q_0\right|)ch(\left|q-q_0\right|)\right)$$

(35)

(6)-(10) sonlarga ko`ra

$$tg\left(q\right)=\frac{1}{\cos^2\left(2\right)+sh^2(\sqrt{50})}\left(\sin(2)\cos(2)+\frac{3i+4j+5k}{\sqrt{50}}sh(\sqrt{50})ch(\sqrt{50})\right)$$

9. $sh\left(q\right)$ **giperbolik sinus kvarternion**

$$sh\left(q_0+q_1i+q_2j+q_3k\right)=\mathrm{Re}(sh(q_0+i\left|q-q_0\right|))+\frac{q-q_0}{\left|q-q_0\right|}\mathrm{Im}(sh(q_0+i\left|q-q_0\right|))$$

(36)

yoki

$$sh\left(q_0+q_1i+q_2j+q_3k\right)=sh(q_0)\cos(\left|q-q_0\right|))+\frac{q-q_0}{\left|q-q_0\right|}ch(q_0)\sin(\left|q-q_0\right|)$$

(37)

(6)-(10) sonlarga ko`ra

$$sh\left(2+3i+4j+5k\right)=sh\left(2\right)\cos(\sqrt{50})+\frac{3i+4j+5k}{\sqrt{50}}ch(2)\sin(\sqrt{50})$$

10. $ch\left(q\right)$ **giperbolik cosinus kvarternion**

$$ch\left(q_0+q_1i+q_2j+q_3k\right)=\mathrm{Re}(ch(q_0+i|q-q_0|))+\frac{q-q_0}{|q-q_0|}\mathrm{Im}(ch(q_0+i|q-q_0|))$$

(38)

yoki

$$ch\left(q_0+q_1i+q_2j+q_3k\right)=ch(q_0)\cos(|q-q_0|))+\frac{q-q_0}{|q-q_0|}sh(q_0)\sin(|q-q_0|)$$

(39)

(6)-(10) sonlarga ko`ra

$$ch\left(2+3i+4j+5k\right)=ch\left(2\right)\cos(\sqrt{50})+\frac{3i+4j+5k}{\sqrt{50}}sh(2)\sin(\sqrt{50})$$

11. $th(q)$ **giperbolik tangens kvarternion**

(12) formuladan foydalanamiz

$$th\left(q_0+q_1i+q_2j+q_3k\right)=\mathrm{Re}(th(q_0+i|q-q_0|))+\frac{q-q_0}{|q-q_0|}\mathrm{Im}(th(q_0+i|q-q_0|)) \quad (40)$$

yoki

$$th\left(q\right)=\frac{1}{ch^2(q_0)+\sin^2(|q-q_0|)}\left(sh(q_0)ch(q_0)+\frac{q-q_0}{|q-q_0|}\sin(|q-q_0|)\cos(|q-q_0|)\right) \quad (41)$$

(6)-(10) sonlarga ko`ra

$$th\left(2+3i+4j+5k\right)=\frac{1}{ch^2(2)+\sin^2(\sqrt{50})}\left(sh(2)ch(2)+\frac{3i+4j+5k}{\sqrt{50}}\sin(\sqrt{50})\cos(\sqrt{50})\right)$$

ADABIYOTLAR:

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