HILFER KASR TARTIBLI OPERATORNI O‘Z ICHIGA OLUVCHI BURGER TENGLAMASI UCHUN CHEGARAVIY MASALA YECHIMINING MAVJUDLIGI HAQIDA

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Annotatsiya.: *Ushbu maqolada kasr tartibli differensial operatorni o‘z ichiga oluvchi Burger tenglamasi uchun chegaraviy masalaning kuchsiz yechimi chegaraviy shartlarda berilgan funksiyalar qanday shartlarni qanoatlantirganda mavjud bo‘lmasligi aniqlangan.*

Kalit so‘zlar. *Burger tenglamasi, Hilfer kasr tartibli differensial operatori, yechimning mavudligi.*

THE EXISTENCE OF A SOLUTION TO A BOUNDARY VALUE PROBLEM FOR THE BURGER'S EQUATION CONTAINING HILFER FRACTIONAL DIFFERENTIAL OPERATOR

Abstract: *In this article, it is determined that the weak solution of the boundary value problem for Burger's equation, which includes a fractional differential operator, does not exist when the functions given in the boundary conditions satisfy the conditions.*

Keywords. *Burger's equation, Hilfer differential operator of fractional order, the existence of solution.*

Kirish. Burger tenglamasi xussusiy hosilali differensial tenglama bo'lib, suyuqlik oqimining harakatini tavsiflaydi. Tenglama birinchi marta uni 1939-yilda taklif qilgan golland matematigi Yoxannes Burger sharafiga nomlangan. Klassik Burger tenglamasi quyidagi ko‘rinishda bo‘ladi:

$$u_t(t,x)+u(t,x)u_x(t,x)=\nu u_{xx}(t,x)$$

bu yerda $u(t,x)$ – suyuqlikning tezligi, t – vaqt, x – pozitsiya va ν - suyuqlikning kinematik yopishqoqligi. Burger tenglamasi chiziqli bo‘lmagan tenglama bo‘lib, u suyuqliklar mexanikasi, transport oqimi va matematik-fizika kabi ko‘plab sohalarda qo‘llaniladi[5].

Mazkur maqolada biz kasr tartibli differensial operatorlarni o‘z ichiga oluvchi Burger tenglamasi uchun chegaraviy masala yechimining mavjud bo‘lmaslik shartlarini aniqlash bilan shug‘ullanamiz.

Asosiy qism. Dastlab, biz kasr hisobning ba’zi asosiy tushuncha va ta’riflarini keltiraylik.

1.1-ta’rif. $f\in L([a,b])$ va $\alpha>0$ bo‘lsin. U holda quyidagi

(1.1) $$I_{a+}^{\alpha}[f](t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)ds$$

va

(1.2) $$I_{b-}^{\alpha}[f](t)=\frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1} f(s)ds$$

integrallar mos holda Riman - Liuvillning chap tomonli va o‘ng tomonli kasr tartibli integrallari deb ataladi, bu yerda $\Gamma(z)$ – Eylerning gamma - funksiyasi.

1.2-ta’rif. Riman - Liuvillning chap tomonli $\alpha(0<\alpha<1)$ kasr tartibli hosilasi $D_{a+}^{\alpha} f$ quyidagicha aniqlanadi :

(1.3) $$D_{a+}^{\alpha}[f](t)=\frac{d}{dt}I_{a+}^{1-\alpha}[f](t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-s)^{-\alpha} f(s)ds.$$

1.3-ta’rif. Riman - Liuvillning o‘ng tomonli $\alpha(0<\alpha<1)$ kasr tartibli hosilasi $D_{b-}^{\alpha} f$ quyidagicha aniqlanadi :

(1.4)
$$D_{b-}^{\alpha}[f](t)=-\frac{d}{dt}I_{b-}^{1-\alpha}[f](t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b (s-t)^{-\alpha} f(s)ds.$$

1.4-ta’rif. Kaputoning $\alpha(n-1<\alpha<n)$ kasr nartibli hosilasi quyidagicha aniqlanadi:

(1.5) $${}_a^C D_t^{\alpha} f(t)=\frac{1}{\Gamma(\alpha-n)}\int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau$$

1.5-ta’rif. Hilferning $0<\alpha<1$ tartibli va $0\le\beta\le1$ tipli hosilasi $D_{a+}^{\alpha,\beta} f$ quyidagicha aniqlanadi:

(1.6) $$D_{a+}^{\alpha,\beta}[f](t)=I_{a+}^{\beta(1-\alpha)}\frac{d}{dt}I_{a+}^{(1-\beta)(1-\alpha)}[f](t),$$

bu yerda I_{a+}^{σ}, $\sigma>0$ – Riman - Liuvill kasr tartibli integrali.

Hilfer hosilasi haqida batafsil ma’lumot uchun [2] , [3] ishlarni tavsiya etamiz. Bu hosila $(\beta=0)$ da Riman - Liuvill hosilasi va $(\beta=1)$ da Kaputo kasr hosilasiga aylanadi [1].

1.1-lemma. $\alpha>0, p\ge1, q\ge1$ va $\frac{1}{p}+\frac{1}{q}\le1+\alpha$ ($\frac{1}{p}+\frac{1}{q}=1+\alpha$ holatda $p\ne1$ va $q\ne1$) bo‘lsin. Agar $\varphi\in L_p(a,b)$ va $\psi\in L_q(a,b)$ bo‘lsa, u holda

(1.7) $$\int_a^b \varphi(t)I_{a+}^{\alpha}[\psi](t)dt=\int_a^b \psi(t)I_{b-}^{a}[\varphi]dt$$

tenglik o‘rinli.

Kasr tartibli operatorni o'z ichiga oluvchi Burger tenglamasi yechimining mavjud emasligi.

$\mathbf{R}^2$ dagi to'g'ri to'rtburchakli sohani $\Pi_{a,b}$ bilan belgilaylik:

$\Pi_{a,b} = \{(t,x) \in \mathbf{R}^2 : 0 < t < T, a < x < b\}.$

$\Pi_{a,b}$ sohada vaqt bo'yicha kasr tartibli ushbu

(2.1) $$D_{0+,t}^{\alpha,\beta} u(t,x) + u(t,x) u_x(t,x) = \nu u_{xx}(t,x)$$

Burger tenglamasi va quyidagi boshlang'ich shart

(2.2) $$I_{0+,t}^{\gamma-1} u(0,x) = u_0(x),\ x \in [a,b]$$

berilgan bo'lsin, bu yerda $D_{a+}^{\alpha,\beta}$ – $0 < \alpha < 1$ tartibli va $0 \le \beta \le 1$ tipli Hilfer hosilasi, $\nu > 0$ va $u_0(x)$ berilgan funksiya.

(2.1) - (2.2) masala yechimining mavjudligi masalasini qaraylik. Buning uchun nochiziqli tenglamalar yechimlarining buzilishini tahlil qilish uchun Poxojayev tomonidan taklif etilgan usuldan foydalanamiz [6].

$T > 0$, $a,b \in \mathbf{R}$ ixtiyoriy parametrlar bilan $\Pi_{a,b}$ sohada aniqlangan $\varphi(t,x)$ funksiyalarning $\Phi(\Pi_{a,b})$ sinfini qaraylik. Bu sinf funksiyalari quyidagi xossalarga ega:

(i) $\varphi_t, \varphi_{xx} \in C(\Pi_{a,b})$;

(ii) $\Pi_{a,b}$ da $\varphi_x \ge 0$;

(iii) $x \in (a,b)$ va $t = T$ da $I_{T-,t}^{\beta(1-\alpha)} \varphi(x,t) = 0$;

(iv) $\zeta(\Pi_{a,b}) = \iint\limits_{\Pi_{a,b}} \frac{(L^* \varphi)^2}{\varphi_x} dtdx < +\infty$;

bu yerda $L^* \varphi = -I_{T-,t}^{(1-\beta)(1-\alpha)} D_{T-,t}^{1-\beta(1-\alpha)} \varphi - \nu \varphi_{xx}$.

Faraz qilaylik, (2.1) - (2.2) masalaning ixtiyoriy $\varphi(x,t) \in \Phi(\Pi_{a,b})$ uchun $u_{xx}, D_{0+,t}^{\alpha,\beta} u \in C([a,b] \times [0,t])$ shartni qanoatlantiradigan kuchsiz yechimi va $T > 0$ son mavjud bo'lsin.

Endi (2.1) tenglamani $\varphi \in \Phi(\Pi_{a,b})$ funksiyaga ko'paytirib, so'ngra $\Pi_{a,b}$ bo'yicha integrallab, quyidagiga ega bo'lamiz:

(2.3) $$\iint\limits_{\Pi_{a,b}} \varphi(t,x) D_{0+,t}^{\alpha,\beta} u(t,x) dtdx + \iint\limits_{\Pi_{a,b}} \varphi(t,x) u(t,x) u_x(t,x) dtdx =$$

$$= \nu \iint\limits_{\Pi_{a,b}} \varphi(t,x) u_{xx}(t,x) dtdx.$$

(2.3) tenglikdagi integrallarda bo'laklab integrallash qoidasidan foydalanib, quyidagi tengliklarni hosil qilamiz:

(2.4) $$\iint_{\Pi_{a,b}} \varphi(t,x)u(t,x)u_x(t,x)dtdx =$$

$$=\frac{1}{2}\int_0^T u^2(t,x)\varphi(t,x)\Big|_a^b dt - \frac{1}{2}\iint_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx,$$

(2.5) $$\iint_{\Pi_{a,b}} \varphi(t,x)u_{xx}(t,x)dtdx =$$

$$=\int_0^T \left[u_x(t,x)\varphi(t,x) - u(t,x)\varphi_x(t,x)\right]\Big|_a^b dt - \iint_{\Pi_{a,b}} u(t,x)\varphi_{xx}(t,x)dtdx.$$

1.5-ta'rif hamda 1.1-lemmadan foydalansak, u holda quyidagi tenglik o'rinli:

$$\iint_{\Pi_{a,b}} \varphi(t,x)D_{0+,t}^{\alpha,\beta}u(t,x)dtdx = \iint_{\Pi_{a,b}} \varphi(t,x)I_{0+}^{\beta(1-\alpha)}\frac{d}{dt}I_{0+}^{(1-\beta)(1-\alpha)}u(t,x)dtdx =$$

$$=\iint_{\Pi_{a,b}} I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)\frac{d}{dt}I_{0+}^{(1-\beta)(1-\alpha)}u(t,x)dtdx.$$

Oxirgi tenglikda bo'laklab integrallash qoidasini qo'llab va 1.1-lemmadan yana bir marta foydalanib, quyidagi tenglikka ega bo'lamiz:

$$\iint_{\Pi_{a,b}} \varphi(t,x)D_{0+,t}^{\alpha,\beta}u(t,x)dtdx = \int_a^b \left\{I_{0+,t}^{(1-\beta)(1-\alpha)}u(t,x)I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)\right\}\Big|_0^T dx -$$

$$-\iint_{\Pi_{a,b}} \frac{d}{dt}I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)I_{0+}^{(1-\beta)(1-\alpha)}u(t,x)dtdx =$$

$$\int_a^b \left\{I_{0+,t}^{(1-\beta)(1-\alpha)}u(t,x)I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)\right\}\Big|_0^T dx - \iint_{\Pi_{a,b}} u(t,x)I_{T-,t}^{(1-\beta)(1-\alpha)}\frac{d}{dt}I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)dtdx.$$

(2.4) va (2.5) dan va 1.3-ta'rifdan foydalansak, (2.3) quyidagi ko'rinishni oladi:

(2.6) $$\frac{1}{2}\iint_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx = \iint_{\Pi_{a,b}} u(t,x)\left(L^*\varphi\right)(t,x) +$$

$$+\int_a^b \left\{I_{0+,t}^{(1-\beta)(1-\alpha)}u(t,x)I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)\right\}\Big|_0^T dx + \int_0^T B\left(u(t,x),\varphi(t,x)\right)\Big|_a^b dt,$$

bu yerda

$$B\left(u(t,x),\varphi(t,x)\right) = \frac{1}{2}u^2(t,x)\varphi(t,x) - \nu u_x(t,x)\varphi(t,x) + \nu u(t,x)\varphi_x(t,x).$$

(2.2) va $\varphi(t,x)$ funksiyaning *(iii)* xossasidan foydalansak, quyidagilarni topamiz:

(2.7) $$\frac{1}{2}\iint_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx = \iint_{\Pi_{a,b}} u(t,x)\left(L^*\varphi\right)(t,x)dtdx +$$

$$+\int_0^T B\big(u(t,x),\varphi(t,x)\big)\Big|_a^b dt-\int_a^b u_0(x)I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)dx.$$

Gyolder va Yung tengsizliklaridan foydalansak,

$$\left|\iint_{\Pi_{a,b}} u(t,x)\big(L^*\varphi\big)(t,x)dtdx\right|=\left|\iint_{\Pi_{a,b}} u(t,x)\sqrt{\varphi_x(x,t)}\frac{\big(L^*\varphi\big)(t,x)}{\sqrt{\varphi_x(x,t)}}dtdx\right|\le$$

$$\left(\iint_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx\right)^{1/2}\left(\iint_{\Pi_{a,b}}\frac{\big(\big(L^*\varphi\big)(t,x)\big)^2}{\varphi_x(x,t)}dtdx\right)^{1/2}\le$$

$$\frac{1}{2}\iint_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx+\frac{1}{2}\iint_{\Pi_{a,b}}\frac{\big(\big(L^*\varphi\big)(t,x)\big)^2}{\varphi_x(x,t)}dtdx$$

munosbatga ega boʻlamiz.

Bu tengsizlikni va *(iv)* xossani e'tiborga olsak, (2.7) quyidagicha koʻrinish oladi:

(2.8) $$0\le\frac{1}{2}\zeta\left(\Pi_{a,b}\right)+\int_0^T B\big(u(t,x),\varphi(t,x)\big)\Big|_a^b dt-\int_a^b u_0(x)I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\Big|_{t=0}dx.$$

Quyidagi teorema oʻrinli:

2.1-teorema. Faraz qilaylik, $u_0(x)\in L[a,b]$ boʻlib, $u_0(x)$ va chegaraviy shartlarda berilgan funksiyalar ushbu shartlarni qanoatlantirsin: shunday $\varphi(x,t)\in\Phi\left(\Pi_{a,b}\right)$ mavjudki, $B\big(u(t,x),\varphi(t,x)\big)\Big|_a^b\in L[0,T]$ boʻlib, quyidagi tengsizlik oʻrinli boʻlsin:

(2.9) $$\frac{1}{2}\zeta\left(\Pi_{a,b}\right)+\int_0^T B\big(u(t,x),\varphi(t,x)\big)\Big|_a^b dt-\int_a^b u_0(x)I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\Big|_{t=0}dx<0.$$

U holda (2.1) - (2.2) masala $u_{xx},D_{0+,t}^{\alpha,\beta}u\in C\big([a,b]\times[0,t]\big)$ yechimi $\Pi_{a,b}$ da mavjud boʻlmaydi.

Isbot. Teskarisidan faraz qilaylik, ya'ni (2.1) – (2.2) masala $\Pi_{a,b}$ da yechimga ega boʻlsin. U holda (2.8) va (2.9) tengsizliklarga koʻra qarama-qarshilikka duch kelamiz. Bundan esa farazimiz notoʻgʻri ekanligi kelib chiqadi.

Misol sifatida $\Pi_{a,b}=\left\{(t,x)\in\mathbf{R}^2:0<t<T,0<x<1\right\}$ toʻgʻri toʻrtburchakli sohada $\nu=1$ boʻlgan (2.1) Burger kasr tartibli tenglamasini (2.2) boshlangʻich shart va

(2.10) $$u(t,0)=\tau_1(t),\ u_x(t,0)=\tau_2(t),\ 0<t<T$$

chegaraviy shartlar bilan qaraylik, bu yerda τ_1 va τ_2 berilgan funksiyalar boʻlib, $\tau_1,\tau_2\in L[0,T]$.

(2.1) vaqt bo'yicha kasr tartibli Burger tenglamasini $\varphi \in \Phi\left(\Pi_{a,b}\right)$ ga ko'paytirganimizdan keying hisoblashlar va soddalashtirishlardan so'ng quyidagiga kelamiz:

$$0 < \frac{1}{2}\zeta\left(\Pi_{0,1}\right) + \int_0^T B\left(u(t,x),\varphi(t,x)\right)\Big|_a^b dt - \int_0^1 u_0(x) I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\big|_{t=0} dx.$$

$\varphi(t,x)$ funksiyaga chegaraviy shartlarni

(2.11) $$\varphi(t,1)=0,\ \varphi_x(t,1)=0,\ 0<t<T.$$

kabi qo'yaylik.

U holda quyidagiga ega bo'lamiz:

$$B(u,\varphi)\Big|_a^b == -\left[\frac{1}{2}\tau_1^2(t)-\tau_2(t)\right]\varphi(t,0)-\tau_1(t)\varphi_x(t,0).$$

Bu holda quyidagi teorema o'rinli:

2.2-teorema. (2.1), (2.2), (2.10) *boshlang'ich chegaraviy masalaning* (2.11) *chegaraviy shartlarni qanoatlantiruvchi* $\varphi \in \Phi\left(\Pi_{0,1}\right)$ *funksiyasi mavjud bo'lib, quyidagi tengsizlik o'rinli bo'lsin:*

(2.12) $$\frac{1}{2}\zeta\left(\Pi_{0,1}\right) < \int_0^T \left[\frac{1}{2}\tau_1^2(t)\varphi(t,0)-\tau_2(t)\varphi(t,0)+\tau_1(t)\varphi_x(t,0)\right]dt + $$

$$+\int_0^1 u_0(x) I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\big|_{t=0} dx.$$

U holda (2.1), (2.2), (2.10) *masala* $\Pi_{0,1}$ *da yechimga ega bo'lmaydi.*

$\Phi\left(\Pi_{a,b}\right)$ funksiyalar sinfi bo'sh emasligiga misol keltiraylik.

2.1-misol. $\varphi(t,x)$ funksiyasi sifatida quyidagi funksiyani olishimiz mumkin:

(2.13) $$\varphi(t,x)=(T-t)^{\delta}(x-1)^3,$$

bu yerda $\delta \in \mathbf{R}$ uchun $\delta > 2\alpha - 1$.

(2.13) funksiya uchun *(i)-(iv)* shartlar bajarilishini osongina ko'rsatish mumkin. Ba'zi hisob kitoblar yordamida ko'rsatish mumkinki:

(2.14) $$\zeta\left(\Pi_{0,1}\right)=\frac{k_1^2}{3}\frac{T^{\delta-2\alpha+1}}{\delta-2\alpha+1}-4k_1\frac{T^{\delta-\alpha+1}}{\delta-\alpha+1}+36\frac{T^{\delta+1}}{\delta+1},$$

bu yerda $k_1 = \Gamma(\delta+1)\Gamma^{-1}(1-\alpha+\delta)$.

2.2- teoremadan quyidagi natijaga kelamiz:

Natija. $u_0 \in L[0,1]$, $\tau_1, \tau_2 \in L[0,T]$ funksiyalar quyidagi tengsizlikni qanoatlantirsin:

$$k_2\int_0^1 u_0(x)(x-1)^3 dx > \int_0^T \left[\frac{1}{2}\tau_1^2(t)-\tau_2(t)-3\tau_1(t)\right](T-t)^{\delta} dt +$$

$$\frac{k_1^2}{6}\frac{T^{\delta-2\alpha+1}}{\delta-2\alpha+1}-2k_1\frac{T^{\delta-\alpha+1}}{\delta-\alpha+1}+18\frac{T^{\delta+1}}{\delta+1}.$$

U holda (2.1), (2.2), (2.10) masala $\Pi_{0,1}$ da yechimga ega boʻlmaydi.

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