# ON ASYMPTOTIC STRUCTURE OF CRITICAL MARKOV BRANCHING PROCESSES WITH POSSIBLY INFINITE VARIANCE

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We consider the Markov Branching Process to be the homogeneous continuous-time Markov chain $\{Z(t), t \ge 0\}$ with state space $\mathcal{S}_0 = \{0\} \cup \mathcal{S}$, where $\mathcal{S} \subset \mathbf{N}$ and $\mathbf{N} = \{1, 2, \ldots\}$. The transition probabilities of the process

$$P_{ij}(t) := \mathbf{P}\{Z(t) = j \mid Z(0) = i\}$$

satisfy the following branching property:

$$P_{ij}(t) = P_{ij}^{i*}(t) \quad \textit{for all} \quad i, j \in \mathcal{S},$$

where the asterisk denotes convolution. Herein transition probabilities $P_{1j}(t)$ are expressed by relation

$$P_{1j}(t) = \delta_{1j} + a_j \varepsilon + o(\varepsilon) \quad \textit{as} \quad \varepsilon \downarrow 0,$$

where $\delta_{ij}$ is Kronecker's delta and $\{a_j\}$ are intensities of individuals' transformation that $a_j \ge 0$ for $j \in \mathcal{S}_0 \setminus \{1\}$ and

$$0 < a_0 < -a_1 = \sum_{j \in \mathcal{S}_0 \setminus \{0\}} a_j < \infty .$$

The process $\{Z(t)\}$ was defined first by Kolmogorov and Dmitriev [4]; for more detailed information see [1, 2].

Defining generating functions $F(t;s) = \sum_{j \in \mathcal{S}_0} P_{1j} s^j$ and $f(s) = \sum_{j \in \mathcal{S}_0} a_j s^j$ for $s \in [0,1)$ we keep on the critical case that is $f'(1-) = 0$ and, assume that the infinitesimal generating function $f(s)$ has the representation

$$f(1 - y) = y\mathcal{L}(y) \qquad [f_\nu]$$

for $y \in (0,1]$ with $\mathcal{L}(y) = y^\nu L(1/y)$, where $0 < \nu \le 1$ and $L(*)$ is slowly varying (SV) function at infinity (in sense of Karamata); see [3, 5]. Note that the function $y\mathcal{L}(y)$ is positive and tends to zero and has a monotone derivative so that $y\mathcal{L}'(y)/\mathcal{L}(y) \to \nu$ as $y \downarrow 0$; see [3, p. 401]. Thence it is natural to write

$$\frac{y\Lambda'(y)}{\Lambda(y)} = \nu + \delta(y), \quad [\Lambda_\delta]$$

where $\delta(y)$ is continuous and $\delta(y) \to 0$ as $y \downarrow 0$.

Let $R(t;s) := 1 - F(t;s)$. Sevastyanov [6] proved that if $f'''(1-) < \infty$ then

$$\frac{1}{R(t;s)} - \frac{1}{1-s} = \frac{f''(1-)}{2}t + O(\ln t)$$

as $t \to \infty$ for all $s \in [0,1)$; see [6, p. 72]. The following lemma is an essentially improvement of Sevastyanov's result.

**Lemma.** *Under the conditions* $[f_\nu]$ *and* $[\Lambda_\delta]$

$$\frac{1}{\Lambda(R(t;s))} - \frac{1}{\Lambda(1-s)} = \nu t + \int_0^t \delta(R(u;s))du.$$

*If in addition* $\delta(y) = \Lambda(y)$ *then*

$$\frac{1}{\Lambda(R(t;s))} - \frac{1}{\Lambda(1-s)} = \nu t + \frac{1}{\nu}\ln \nu(t;s) + o(\ln \nu(t;s))$$

*as* $t \to \infty$, *where* $\nu(t;s) = \Lambda(1-s)\nu t + 1$.

From this Lemma we obtain the following two limit theorems.

**Theorem 1.** *Let conditions* $[f_\nu]$, $[\Lambda_\delta]$ *hold and* $\delta(y) = \Lambda(y)$. *Then*

$$\mathbf{P}\{Z(t) > 0\} = \frac{\mathcal{N}(t)}{(\nu t)^{1/\nu}}\left(1 - \frac{\ln[a_0\nu t + 1]}{\nu^3 t} + o\left(\frac{\ln t}{t}\right)\right)$$

*as* $t \to \infty$, *where* $\mathcal{N}(t)$ *is the SV-function satisfying a condition*

$$\mathcal{N}^{\nu}(t)\cdot\Lambda\left(\frac{(\nu t)^{1/\nu}}{\mathcal{N}(t)}s\right) \sim s, \quad s \to 1, \quad as\ t \to \infty.$$

**Theorem 2.** *Let conditions* $[f_\nu]$, $[\Lambda_\delta]$ *hold and* $\delta(y) = \Lambda(y)$. *Then*

$$(\nu t)^{1+1/\nu}P_{11}(t) = \frac{\mathcal{N}(t)}{a_0}\left(1 - \frac{1+\nu}{\nu^3}\frac{\ln[a_0\nu t + 1]}{t} + o\left(\frac{\ln t}{t}\right)\right)$$

*as* $t \to \infty$, *where the SV-function* $\mathcal{N}(t)$ *is defined in Theorem 1.*

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