ON GRADIENT GIBBS MEASURES WITH 4-PERIODIC BOUNDARY LAWS OF MODEL OF SOS TYPE ON THE CAYLEY TREE OF ORDER TWO AND THREE

Ilyasova R. A
Karimova S. A
National University of Uzbekistan, Tashkent, Uzbekistan e-mail
ilyasova.risolat@mail.ru, sevinch0603@mail.ru

We consider Gradient Gibbs measures corresponding to a periodic boundary law for a generalized SOS model with spin values from a countable set, on Cayley trees. On the Cayley tree, detailed information on Gradient Gibbs measures for models of SOS model are given in [3, 8, 11, 16]. Investigating these works for the generalized SOS model, in this paper the problem of finding Gradient Gibbs measures which correspond to periodic boundary laws is reduced to a functional equation.

By solving this equation all Gradient Gibbs measures with 4 periodic boundary laws are found.

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INTRODUCTION

The gradient Gibbs measure is a probability measure on the space of gradient fields defined on a manifold. It is often used in statistical mechanics to describe the equilibrium states of a system. The gradient Gibbs measure is derived from the Gibbs measure, which is a probability measure on the space of field configurations. The critical difference is that the gradient Gibbs measure focuses on the gradients of the fields rather than the fields themselves (e.g. [7]). Specifically, the Gradient Gibbs measure is defined on the set of spin configurations of a system on a Cayley tree. The Gradient Gibbs measure on a Cayley tree assigns a probability to each possible spin configuration based on the energy of that configuration. The energy of a spin configuration is determined by the interactions between neighboring spins. In the case of a Cayley tree, each spin is coupled to its nearest neighbors along the edges of the tree (see [5]).

Mathematically, the gradient Gibbs measure assigns a probability to each possible configuration of a gradient field on the Cayley tree, based on an energy function. The energy function typically represents the interactions between the gradients of a scalar field or a vector field. The probability of a configuration is proportional to the exponential of the negative energy of that configuration (e.g. [1, 4, 5, 12, 14]).

The study of random field ξ_x from a lattice graph (e.g., $\mathbb{Z}^d$ or a Cayley tree Γ^k) to a measure space (E, E) is a central component of ergodic theory and statistical physics. In many classical models from physics (e.g., the Ising model, the Potts model, the SOS model), E is a finite set (i.e., with a finite underlying measure λ), and ξ_x has a physical interpretation as the spin of a particle at location x in a crystal lattice (detail in [1, 2, 3, 6, 7, 8, 9, 10, 14, 15]).

Let us give basic definitions and some known facts related to (gradient) Gibbs measures. The Cayley tree $\Gamma^k = (V, L)$ of order k ≥ 1 is an infinite tree, i.e. connected and undirected graph without cycles, each vertex of which has exactly k + 1 edges. Here V is the set of vertices of Γ^k and L is the set of its edges.

Consider models where the spin takes values in the set $\Phi \subseteq \mathbb{R}^+_\infty$, and is assigned to the vertices of the tree. Let $\Omega_A = \Phi^A$ be the set of all configurations on A and $\Omega := \Phi^V$. A partial order ≤ on Ω defined pointwise by stipulating that σ1 ≤ σ2 if and only if σ1(x) ≤ σ2(x) for all x ϵ V . Thus (Ω;≤) is a poset, and whenever we consider Ω as a poset then it will always be with respect to this partial order. The poset Ω is complete. Also, Ω can be considered as a metric space with respect to the metric $\rho : \Omega \times \Omega \to \mathbb{R}^+$ given by

$$\rho\left(\{\sigma(x_n)\}_{x_n \in V}, \{\sigma'(x_n)\}_{x_n \in V}\right) = \sum_{n \geq 0} 2^{-n} \mathrm{X}_{\sigma(x_n) \neq \sigma'(x_n)},$$

where $V = \{x_0, x_1, x_2, ...\}$ and X_A is the indicator function.

We denote by N the set of all finite subsets of V . For each $A \in V$ let $\pi_A : \Omega \to \Phi^A$ be given by $\pi_A(\sigma_x)_{x\in V} = (\sigma_x)_{x \in A}$ and let $\mathrm{C}_A = \pi_A^{-1}\left(\mathrm{P}\left(\Phi^A\right)\right)$. Let $\mathrm{C} = \bigcup_{A \in \mathrm{N}} \mathrm{C}_A$ and F is the smallest sigma field containing C. Write $\mathrm{T}_\Lambda = \mathrm{F}_{V \setminus \Lambda}$ and T for the tail-σ-algebra, i.e., intersection of T_Λ over all finite subsets Λ of L: The sets in T are called tail-measurable sets.

Definition 1. [5] Let $P_\Lambda : \Omega \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ be F_Λ-measurable mapping for all Λ ϵ N, then the collection $P = \{P_\Lambda\}_{\Lambda \in \mathrm{N}}$ is called a potential. Also, the following expression

$$H_{\Delta,P}(\sigma) \overset{\text{def}}{=} \sum_{\Delta \cap \Lambda \neq \varnothing, \Lambda \in \mathrm{N}} P_\Delta(\sigma), \quad \forall \sigma \in \Omega. \tag{1}$$

is called Hamiltonian H associated with the potential P.

For a fixed inverse temperature β > 0, the Gibbs specification is determined by a family of probability kernels $\zeta = (\zeta_\Lambda)_{\Lambda \in \mathrm{N}}$ defined on $\Omega_\Lambda \times \mathrm{F}_{\Lambda^c}$ by the Boltzmann-Gibbs weights

$$\zeta_\Lambda(\sigma_\Lambda | \omega) = \frac{1}{\mathbf{Z}_\Lambda^\omega} e^{-\beta H^\omega_{\Lambda,P}(\sigma_\Lambda \omega)} \tag{2}$$

where $\mathbf{Z}_\Lambda^\omega = \sum_{\sigma \in \Omega_\Lambda} e^{-\beta H^\omega_{\Lambda,P}(\sigma_\Lambda)}$ is the partition function, related to free energy. From [5], the family of mappings $\{\zeta_\Lambda(\sigma | \omega)\}_{\Lambda \in \mathrm{N}}$ is the family of proper F_Λ- measurable quasi-probability kernels. Thus, the collection $\mathrm{V} = \{\zeta_\Lambda\}_{\Lambda \in \mathrm{N}}$ will be called an F-specification if $\zeta_\Delta = \zeta_\Delta \zeta_\Lambda$ whenever $\Lambda, \Delta \in \mathrm{N}$ with $\Lambda \subseteq \Delta$. Let $\mathrm{V} = \{\zeta_\Lambda\}_{\Lambda \in \mathrm{N}}$ be an F-specification; then a probability measure μ ϵ P(F) is called a Gibbs measure with specification V if $\mu = \mu\zeta_{\Lambda}$ for each $\Lambda \in \mathrm{N}$.

1 Gradient Gibbs measure

For any configuration $\omega = (\omega(x))_{x\in V} \in \mathbb{Z}^{V}$ and edge $e = \langle x, y\rangle$ of $\vec{L}$ (oriented) the difference along the edge e is given by $\nabla\omega_e = \omega_y - \omega_x$ and $\nabla\omega$ is called the gradient field of ω. The gradient spin variables are now defined by $\eta_{\langle x,y\rangle} = \omega_y - \omega_x$ for each $\langle x, y\rangle$. The space of gradient configurations is denoted by Ω^{∇}. The measurable structure on the space Ω^{∇} is given by σ-algebra

$$\mathrm{F}^{\nabla} := \sigma\left(\{\eta_e | \, e \in \vec{L}\}\right).$$

Note that F^{∇} is the subset of F containing those sets that are invariant under translation $\omega \to \omega + c$ for $c \in E$. Similarly, we define

$$\mathrm{T}_{\Lambda}^{\nabla} = \mathrm{T}_{\Lambda} \cap \mathrm{F}^{\nabla}, \mathrm{F}_{\Lambda}^{\nabla} = \mathrm{F}_{\Lambda} \cap \mathrm{F}^{\nabla}$$

For nearest-neighboring (n.n.) interaction potential $\Phi = \left(\Phi_b\right)_b$, where $b = \langle x, y\rangle$ is an edge, define symmetric transfer matrices Q_b by

$$Q_b\left(\omega_b\right) = e^{-\left(\Phi_b(\omega_b) + |\partial x|^{-1}\Phi_{\{x\}}(\omega_x) + |\partial y|^{-1}\Phi_{\{y\}}(\omega_y)\right)}.$$

Define the Markov (Gibbsian) specification as

$$\gamma_{\Lambda}^{\Phi}\left(\sigma_{\Lambda} = \omega_{\Lambda} | \, \omega\right) = \left(Z_{\Lambda}^{\Phi}\right)(\omega)^{-1} \prod_{b\cap\Lambda\neq 0} Q_b\left(\omega_b\right).$$

If for any bond $b = \langle x, y\rangle$ the transfer operator $Q_b\left(\omega_b\right)$ is a function of gradient spin variable $\zeta_b = \omega_y - \omega_x$ then the underlying potential Φ is called a gradient interaction potential. Note that for all $A \in \mathrm{F}^{\nabla}$, the kernels $\gamma_{\Lambda}^{\Phi}(A,\omega)$ are F^{∇} -measurable functions of ω, it follows that the kernel sends a given measure μ on $\left(\Omega, \mathrm{F}^{\nabla}\right)$ to another measure $\mu\gamma_{\Lambda}^{\Phi}$ on $\left(\Omega, \mathrm{F}^{\nabla}\right)$. A measure μ on $\left(\Omega, \mathrm{F}^{\nabla}\right)$ is called a gradient Gibbs measure if it satisfies the equality $\mu\gamma_{\Lambda}^{\Phi} = \mu$ (detail in [10, 11, 13]).

Note that, if μ is a Gibbs measure on (Ω, F), then its restriction to F^{∇} is a gradient Gibbs measure. A boundary law is called q-periodic if $l_{xy}(i+q) = l_{xy}(i)$ for every oriented edge $\langle x, y\rangle \in \vec{L}$ and each $i \in \mathbb{Z}$.

It is known that there is a one-to-one correspondence between boundary laws and tree indexed Markov chains if the boundary laws are normalisable in the sense of Zachary [15]:

Definition 2. (Normalisable boundary laws). A boundary law l is said to be normalisable if and only if

$$\sum_{\omega_x\in\mathbb{Z}}\left(\prod_{z\in\partial x}\sum_{\omega_z\in\mathbb{Z}} Q_{zx}\left(\omega_x, \omega_z\right) l_{zx}\left(\omega_z\right)\right) < \infty$$

for any $x \in V$.

The correspondence now reads the following:

Theorem 1. (Theorem 3.2 in [15]). For any Markov specification γ with associated family of transfer matrices $(Q_b)_{b\in L}$ we have

1.Each normalisable boundary law $(l_{xy})_{x,y}$ for $(Q_b)_{b\in L}$ defines a unique tree-indexed Markov chain $\mu \in \mathrm{G}(\gamma)$ via the equation given for any connected set $\Lambda \in \mathrm{S}$

$$\mu\left(\sigma_{\Lambda\cup\partial\Lambda} = \omega_{\Lambda\cup\partial\Lambda}\right) = \left(Z_\Lambda\right)^{-1} \prod_{y\in\partial\Lambda} l_{yy_\Lambda}\left(\omega_y\right) \prod_{b\cap\Lambda\neq\varnothing} Q_b\left(\omega_b\right) \qquad (3)$$

where for any $y \in \partial\Lambda, y_\Lambda$ denotes the unique n.n. of y in Λ.

2. Conversely, every tree-indexed Markov chain $\mu \in \mathrm{G}(\gamma)$ admits a representation of the form (3.15) in terms of a normalisable boundary law (unique up to a constant positive factor).

The Markov chain μ defined in (3) has the transition probabilities

$$P_{xy}(i,j) = \mu\left(\sigma_y = j \mid \sigma_x = i\right) = \frac{l_{yx}(j)Q_{yx}(j,i)}{\sum_s l_{yx}(s)Q_{yx}(s,i)} \qquad (4)$$

The expressions (4) may exist even in situations where the underlying boundary law $(l_{xy})_{x,y}$ is not normalisable. However, the Markov chain given by (4), in general, does not have an invariant probability measure. Therefore in [8]; [9]; [10]; [11] some nonnormalisable boundary laws are used to give gradient Gibbs measures.

Now we give some results of above-mentioned paper. Consider a model on Cayley tree $\Gamma^k = (V, \vec{L})$, where the spin takes values in the set of all integer numbers $\mathbb{Z}$. The set of all configurations is $\Omega := \mathbb{Z}^V$.

Now we consider the following Hamiltonian:

$$H(\sigma) = -J \sum_{\langle x,y\rangle} \alpha(|\sigma_x - \sigma_y|)\,|\sigma_x - \sigma_y|, \qquad (5)$$

where

$$\alpha(|m|) = \begin{cases} p_1, & \text{if \$m} \in 2\mathbb{Z}\text{\$} \\ p_2, & \text{if \$m} \in 2\mathbb{Z}+1\text{\$} \end{cases}, \quad p_1, p_2 \in \mathbb{R}^+.$$

Note that if $p_1 = p_2$ then the considered model is called SOS model.

For the Hamiltonian (5) the transfer operator is defined by

$$Q(i,j) = e^{-J\beta\alpha(|i-j|)|i-j|},$$

where β > 0 is the inverse temperature and $J \in \mathbb{R}$.

Also, the boundary law equation of the Hamiltonian can be written as:

$$z_i = \left(\frac{Q(i,0) + \sum_{j\in Z_0} Q(i,j) z_j}{Q(0,0) + \sum_{j\in Z_0} Q(0,j) z_j} \right)^k. \qquad (6)$$

Put $\theta := \exp(-J\beta) < 1$. For translation invariant boundary law, the transfer operator Q reads $Q(i-j) = \theta^{|i-j|}$ for any $i, j \in \mathbb{Z}$. If $\theta := e^{-J\beta} < 1$ then we can write the equation (6) as

$$z_i=\left(\frac{\theta^{\alpha(|i|)|i|}+\sum_{j\in Z_0}\theta^{\alpha(|i-j|)|i-j|}z_j}{1+\sum_{j\in Z_0}\theta^{\alpha(|j|)|j|}z_j}\right)^k,\ i\in\mathbb{Z}_0:=\mathbb{Z}\setminus\{0\}. \qquad (7)$$

Let $\{z_i\}_{i\in\mathbb{Z}}$ be q-periodic sequence, i.e. $z_i=z_{i+q}$ for all $i\in\mathbb{Z}$.

$$\begin{cases} z_1=\left(\dfrac{\theta^{\alpha(|1|)}+\sum_{j\in Z_0}\theta^{\alpha(|1-j|)|1-j|}z_j}{1+\sum_{j\in Z_0}\theta^{\alpha(|j|)|j|}z_j}\right)^k; \\ z_2=\left(\dfrac{\theta^{2\alpha(|2|)}+\sum_{j\in Z_0}\theta^{\alpha(|2-j|)|2-j|}z_j}{1+\sum_{j\in Z_0}\theta^{\alpha(|j|)|j|}z_j}\right)^k; \\ \dots\quad\dots\quad\dots\quad\dots\quad\dots \\ z_q=\left(\dfrac{\theta^{q\alpha(|q|)}+\sum_{j\in Z_0}\theta^{\alpha(|q-j|)|q-j|}z_j}{1+\sum_{j\in Z_0}\theta^{\alpha(|j|)|j|}z_j}\right)^k. \end{cases} \qquad (8)$$

Proposition 1. Let $\{z_i\}_{i\in\mathbb{Z}}$ be q-periodic sequence. Then finding q-periodic solutions
to the system (7) is equivalent to solving the system of equations (8).

Proof. To prove the Proposition, it is sufficient to show $z_i=z_{q+i}$ for all $i\in\{1,2,\dots,q-1,q\}$. Since $z_0=0$, for a fixed $i_0\in\mathbb{Z}$, the numerator of the fraction in (7) can be written as

$$\theta^{\alpha(|i_0|)|i_0|}+\sum_{j\in\mathbb{Z}_0}\theta^{\alpha(|i_0-j|)|i_0-j|}z_j=\sum_{j\in\mathbb{Z}}\theta^{\alpha(|i_0-j|)|i_0-j|}z_j.$$

Also, it can be rewritten as

$$\sum_{j\in\mathbb{Z}}\theta^{\alpha(|i_0-j|)|i_0-j|}z_j=\dots+\theta^{2p_1}z_{i_0-2}+\theta^{p_2}z_{i_0-1}+z_{i_0}+\theta^{p_2}z_{i_0+1}+\theta^{2p_1}z_{i_0+2}+\dots \qquad (9)$$

Similarly, for i_0+q we have

$$\sum_{j\in\mathbb{Z}}\theta^{\alpha(|i_0+q-j|)|i_0+q-j|}z_j=\dots+\theta^{2p_1}z_{i_0+q-2}+\theta^{p_2}z_{i_0+q-1}+z_{i_0+q}+\theta^{p_2}z_{i_0+q+1}+\theta^{2p_1}z_{i_0+q+2}+\dots \qquad (10)$$

If we change z_{k+q} in (10) to z_k for all $k\in\mathbb{Z}$ then we obtain (9). Namely, we have proved

$$z_{i_0}=\left(\frac{\theta^{i_0\alpha(|i_0|)}+\sum_{j\in Z_0}\theta^{\alpha(|i_0-j|)|i_0-j|}z_j}{1+\sum_{j\in Z_0}\theta^{\alpha(|j|)|j|}z_j}\right)^k=\left(\frac{\theta^{(i_0+q)\alpha(|i_0+q|)}+\sum_{j\in Z_0}\theta^{\alpha(|i_0+q-j|)|i_0+q-j|}z_j}{1+\sum_{j\in Z_0}\theta^{\alpha(|j|)|j|}z_j}\right)^k=z_{i_0+q}.$$

Let $u_i=u_0\sqrt[k]{z_i}$ for some $u_0>0$. Then using the Proposition 1 we obtain

$$u_i=\frac{\dots+\theta^{3p_2}u_{i-3}^k+\theta^{2p_1}u_{i-2}^k+\theta^{p_2}u_{i-1}^k+u_i^k+\theta^{p_2}u_{i+1}^k+\theta^{2p_1}u_{i+2}^k+\theta^{3p_2}u_{i+3}^k+\dots}{\dots+\theta^{3p_2}u_{-3}^k+\theta^{2p_1}u_{-2}^k+\theta^{p_2}u_{-1}^k+u_0^k+\theta^{p_2}u_1^k+\theta^{2p_1}u_2^k+\theta^{3p_2}u_3^k+\dots}.$$

We can rewrite the last system of equations in the following form:

$$u_i=\frac{\sum_{j=1}^{\infty}\theta^{2p_1 j}u_{i-2j}^k+\sum_{j=1}^{\infty}\theta^{(2j-1)p_2}u_{i-2j+1}^k+u_i^k+\sum_{j=1}^{\infty}\theta^{(2j-1)p_2}u_{i+2j-1}^k+\sum_{j=1}^{\infty}\theta^{2p_1 j}u_{i+2j}^k}{\sum_{j=1}^{\infty}\theta^{2p_1 j}u_{-2j}^k+\sum_{j=1}^{\infty}\theta^{(2j-1)p_2}u_{-2j+1}^k+u_0^k+\sum_{j=1}^{\infty}\theta^{(2j-1)p_2}u_{2j-1}^k+\sum_{j=1}^{\infty}\theta^{2p_1 j}u_{2j}^k},\quad \text{where}\quad i\in\mathbb{Z}.$$

(11)

2 Main results

In this section, we find periodic solutions (defined in [16]) to (7) which correspond to periodic boundary condition. Namely, for all m ϵ Z we consider the following sequence:

$$u_n=\begin{cases}1, if n=2m;\\ a, if n=4m-1;\\ b, if n=4m+1,\end{cases}\qquad (12)$$

where a and b are some positive numbers.

By Proposition 1, finding solutions that are formed in (12) to (7) is equivalent to solving the following system of equations:

$$\begin{cases} a=\dfrac{\ldots+\theta^{4p_1}a^k+\theta^{3p_2}+\theta^{2p_1}b^k+\theta^{p_2}+a^k+\theta^{p_2}+\theta^{2p_1}b^k+\theta^{3p_2}+\theta^{4p_1}a^k+\ldots}{\ldots+\theta^{4p_1}+\theta^{3p_2}b^k+\theta^{2p_1}+\theta^{p_2}a^k+1+\theta^{p_2}b^k+\theta^{2p_1}+\theta^{3p_2}a^k+\theta^{4p_1}+\ldots};\\ b=\dfrac{\ldots+\theta^{4p_1}b^k+\theta^{3p_2}+\theta^{2p_1}a^k+\theta^{p_2}+b^k+\theta^{p_2}+\theta^{2p_1}a^k+\theta^{3p_2}+\theta^{4p_1}b^k+\ldots}{\ldots+\theta^{4p_1}+\theta^{3p_2}b^k+\theta^{2p_1}+\theta^{p_2}a^k+1+\theta^{p_2}b^k+\theta^{2p_1}+\theta^{3p_2}a^k+\theta^{4p_1}+\ldots}.\end{cases}\qquad (13)$$

Namely,

$$\begin{cases} a=\dfrac{2(\theta^{p_2}+\theta^{3p_2}+\ldots)+(1+2\theta^{4p_1}+2\theta^{8p_1}+\ldots)a^k+2(\theta^{2p_1}+\theta^{6p_1}+\ldots)b^k}{1+2\theta^{2p_1}+2\theta^{4p_1}+\ldots+(\theta^{p_2}+\theta^{3p_2}+\ldots)(a^k+b^k)};\\ b=\dfrac{2(\theta^{p_2}+\theta^{3p_2}+\ldots)+(1+2\theta^{4p_1}+2\theta^{8p_1}+\ldots)b^k+2(\theta^{2p_1}+\theta^{6p_1}+\ldots)a^k}{1+2\theta^{2p_1}+2\theta^{4p_1}+\ldots+(\theta^{p_2}+\theta^{3p_2}+\ldots)(a^k+b^k)}\end{cases}\qquad (14)$$

Taking into account θ < 1 one writes the last system of equations as follows:

$$a=\frac{\frac{2\theta^{p_2}}{1-\theta^{2p_2}}+\frac{1+\theta^{4p_1}}{1-\theta^{4p_1}}a^k+\frac{2\theta^{2p_1}}{1-\theta^{4p_1}}b^k}{\frac{1+\theta^{2p_1}}{1-\theta^{2p_1}}+\frac{\theta^{p_2}}{1-\theta^{2p_2}}(a^k+b^k)},\quad b=\frac{\frac{2\theta^{p_2}}{1-\theta^{2p_2}}+\frac{1+\theta^{4p_1}}{1-\theta^{4p_1}}b^k+\frac{2\theta^{2p_1}}{1-\theta^{4p_1}}a^k}{\frac{1+\theta^{2p_1}}{1-\theta^{2p_1}}+\frac{\theta^{p_2}}{1-\theta^{2p_2}}(a^k+b^k)}.\qquad (15)$$

For all $k\in\mathbb{N}$ and $p_1,p_2\in\mathbb{N}$, the analysis of (15) is so difficult and that's why we consider the case $\frac{1}{p_1}=p_2=2$ and $k=2$. Then (15) can be written as

$$a=\frac{\tau^2a^2+2\tau b^2+2}{a^2+b^2+\tau(\tau+2)},\quad b=\frac{2\tau a^2+\tau^2b^2+2}{a^2+b^2+\tau(\tau+2)}\qquad (16)$$

where $\tau=\theta+\frac{1}{\theta}>2.$

At first, we consider the case a = b. The system of equations (16) is reduced to the polynomial equation:

$$2a^3-\tau(\tau+2)a^2+\tau(\tau+2)a-2=0.\qquad (17)$$

Since the last equation has a solution a = 1, we divide both sides of (17) by a - 1. Consequently, one gets

$$2a^2-(\tau^2+2\tau-2)a+2=0.$$

For any value of the parameter τ the last quadratic equation has two solutions.

Theorem 2. Let $\tau = J\beta + \frac{1}{J\beta}$.Then for the model (5) on the the Cayley tree of order two the following assertion holds:

1. For any value of the parameter τ there are precisely three GGMs associated with a 2-periodic boundary law.

Let k = 3: In this case, in order to find 2-periodic boundary law we will consider the following equation:

$$a^4-\psi a^3+\psi a-1=0 \qquad (18)$$

where $\psi = \frac{\tau(\tau+2)}{2}$.

Since the last equation has a solution a = 1, we divide both sides of (17) by a - 1. Consequently, one gets

$$(a+1)(a^2-\psi a+1)=0.$$

For any value of the parameter τ the last quadratic equation has two solutions.

Theorem 3. Let $\tau = J\beta + \frac{1}{J\beta}$. Then for the model (5) on the the Cayley tree of order three the following assertion holds:

1. For any value of the parameter τ there are precisely three GGMs associated with a 2-periodic boundary law.

It is important to consider Gradient Gibbs measures associated with a 4-periodic boundary law for the model (5). The following theorem gives us a full description of Gradient Gibbs measures associated with a 4-periodic boundary law.

Theorem 4. Let $\tau = J\beta + \frac{1}{J\beta}$, $\tau_{cr}^{(1)} \approx 3.22$.Then for Gradient Gibbs measures associated with a 4-periodic boundary law for the model (5) on the Cayley tree of order two the following statements hold:

1. If $\tau < \tau_{cr}^{(1)}$, then there are three GGMs associated with a 4-periodic boundary law.
2. If $\tau = \tau_{cr}^{(1)}$, then there are five such GGMs.
3. If $\tau > \tau_{cr}^{(1)}$, then there are exactly seven such GGMs. In each case one of solutions is a = b = 1.

Proof. Now we consider the case a 6= b. Then the system of equations (16) can be written as

$$\begin{cases} a^3+ab^2+\tau(\tau+2)a=\tau^2a^2+2\tau b^2+2\,; \\ b^3+a^2b+\tau(\tau+2)b=\tau^2b^2+2\tau a^2+2. \end{cases} \qquad (19)$$

Now we subtract the second equation of (19) from the first one and get

$$a^3 - b^3 - ab(a-b) + \tau(\tau+2)(a-b) = \tau(\tau-2)(a^2-b^2).$$

Since a ≠ b, both sides can be divided by a - b and one gets

$$a^2 + b^2 + \tau(\tau+2) = \tau(\tau-2)(a+b). \qquad (20)$$

By adding the second and first equations of (19), we have

$$(a+b)(a^2+b^2) + \tau(\tau+2)(a+b) = \tau(\tau+2)(a^2+b^2) + 4 \qquad (21)$$

Let a+b = x and ab = y. By using (20) and (21) one gets a new system of equations with respect to x and y that is equivalent to (19):

$$\begin{cases} x^2 - 2y + \tau(\tau+2) = \tau(\tau-2)x \\ x^3 - 2xy + \tau(\tau+2)x = \tau(\tau+2)(x^2-2y) + 4 \end{cases} \qquad (22)$$

In order to find the number of solutions of the last system we can consider the following quadratic equation with respect to x:

$$\tau(\tau-2)x^2 - \tau^2(\tau^2-4)x + \tau^2(\tau+2)^2 - 4 = 0. \qquad (23)$$

It is easy to check that

$$x_1 = \frac{\tau^2(\tau^2-4) + \sqrt{\tau(\tau-2)(\tau^6+2\tau^5-8\tau^4-24\tau^3-16\tau^2+16)}}{2\tau(\tau-2)}$$

and

$$x_2 = \frac{\tau^2(\tau^2-4) - \sqrt{\tau(\tau-2)(\tau^6+2\tau^5-8\tau^4-24\tau^3-16\tau^2+16)}}{2\tau(\tau-2)}$$

are solutions to the equation (23). Put $P(\tau) = \tau^6 + 2\tau^5 - 8\tau^4 - 24\tau^3 - 16\tau^2 + 16$. From τ > 2 it is sufficient to find only positive roots of P(τ). By Descartes' theorem (e.g. [12]) P(τ) has at most two positive roots. Now we find the first derivative of P(τ) and get

$$Q(\tau) = 2\tau(3\tau^4 + 5\tau^3 - 16\tau^2 - 36\tau - 16)$$

By Descartes' theorem Q(τ) has at most one positive root. Using Q(2) < 0 and Q(3) > 0 i.e., by Intermediate Value Theorem we can conclude Q(τ) has at least one root in the segment [2; 3].

On the other hand, P(0) > 0 and P(1) < 0 i.e., by Intermediate Value Theorem P(τ) has one root in the segment [0; 1]: Hence, P(τ) has exactly one positive root which belongs to the interval $(2,\infty)$. Let $\tau_{cr} (\tau_{cr}^{(1)} \approx 3.22)$ be the positive root of the polynomial. Consequently, we can conclude that if $2 < \tau < \tau_{cr}^{(1)}$ then the system of equations (22) has not any positive solution. Let $\tau = \tau_{cr}^{(1)}$, then the system (22) has exactly one positive root. For the case $\tau > \tau_{cr}^{(1)}$, then (22) has exactly two positive roots if we can show $x_1 > 0$. Namely, after short calculations, $\tau > \tau_{cr}^{(1)}$ then we can show the inequality

$$\frac{\tau^2(\tau^2-4) - \sqrt{\tau(\tau-2)(\tau^6+2\tau^5-8\tau^4-24\tau^3-16\tau^2+16)}}{2\tau(\tau-2)} > 0$$

is equivalent to the inequality $\tau^2 + 2\tau - 2 > 0$.

For the case $\tau > \tau_{cr}^{(1)}$, from $a+b=x_i$ and $ab=y_i$ ($i \in \{1,2\}$) after short calculations, we have two quadratic equations respectively to x_1 and x_2:

$$a^2 - x_1 a - \frac{(\tau^2-2\tau-1)x_1 - \tau^2 - 2\tau}{2} = 0$$

and

$$a^2 - x_2 a - \frac{(\tau^2-2\tau-1)x_2 - \tau^2 - 2\tau}{2} = 0.$$

The discriminants are

$$D_{1,2}(\tau) = \frac{3\tau^6 - 2\tau^5 - 20\tau^4 + 16\tau^2 + 8 \pm (3\tau^2 - 2\tau - 2)\sqrt{\tau(\tau-2)(\tau^6 + 2\tau^5 - 8\tau^4 - 24\tau^3 - 16\tau^2 + 16))}}{2\tau(\tau-2)}.$$

Now we find positive zeroes of

$$3\tau^6 - 2\tau^5 - 20\tau^4 + 16\tau^2 + 8 \pm (3\tau^2 - 2\tau - 2)\sqrt{\tau(\tau-2)(\tau^6 + 2\tau^5 - 8\tau^4 - 24\tau^3 - 16\tau^2 + 16))} = 0.$$

A solution to the last equation is also the solution to the following equation:

$$3\tau^8 + 16\tau^7 + 4\tau^6 + 120\tau^5 - 176\tau^4 - 128\tau^3 + 112\tau^2 + 32\tau + 16 = 0.$$

By Descartes' theorem $R(\tau) = 3\tau^8 + 16\tau^7 + 4\tau^6 + 120\tau^5 - 176\tau^4 - 128\tau^3 + 112\tau^2 + 32\tau + 16$ have at most two positive roots. Since R(0) > 0, R(1) < 0; R(2) > 0 and $\lim_{\tau\to\infty} R(\tau) = +\infty$ we can conclude they are not in the interval $(2,\infty)$. Consequently, $D_{1,2}(\tau) > 0$ for any value of $\tau \in (\tau_{cr}^{(1)}, \infty)$

Finally, we consider the case $\tau = \tau_{cr}^{(1)}$. From above, it is sufficient to solve the following equation:

$$4a^2 - 2\tau(\tau+2)a - \tau(\tau+1)(\tau+2)(\tau-3) = 0.$$

Its discriminant is

$$D(\tau) = 4\tau(\tau+2)(5\tau^2 - 6\tau - 24).$$

It's easy to check $D(\tau_{cr}^{(1)}) > 0$, thus there are two positive solutions to (19).

Theorem 5. Let $\tau = J\beta + \frac{1}{J\beta}$, $\tau_{cr}^{(2)} \approx 2.26$.Then for the parameter $\tau \in (2, \tau_{cr}^{(2)})$ there are not any Gradient Gibbs measures associated with a 4-periodic boundary law satisfying the equality a ≠ b for the model (5) on the Cayley tree of order three.

Proof. Now we consider the case a ≠ b. Then the system of equations (16) can be written as

$$\begin{cases} a^4 + ab^3 + \tau(\tau+2)a = \tau^2 a^3 + 2\tau b^3 + 2; \\ b^4 + a^3 b + \tau(\tau+2)b = \tau^2 b^3 + 2\tau a^3 + 2. \end{cases} \qquad (24)$$

In this case, applying the same solution above one gets system of equations with respect to x and y that is equivalent to (24):

$$\begin{cases} x^4 - 3x^2 y + \tau(\tau+2)x = \tau(\tau+2)(x^3 - 3xy) + 4 \\ x^3 - 3xy + \tau(\tau+2) = \tau(\tau-2)(x^2 - y) \end{cases} \qquad (25)$$

In order to find the number of solutions of the last system we can consider the following quartic equation with respect to x:

$$\tau(\tau-2)x^4-\tau^2(\tau-2)(\tau+2)x^3+(\tau^4+6\tau^3+8\tau^2-6)x+2\tau(\tau-2)=0. \qquad (26)$$

Finding the first and second derivatives of $R(x,\tau)=\tau(\tau-2)x^4-\tau^2(\tau-2)(\tau+2)x^3+(\tau^4+6\tau^3+8\tau^2-6)x+2\tau(\tau-2)=0$ we get following two polynomials:

$$S(x,\tau)=4\tau(\tau-2)x^3-3\tau^2(\tau-2)(\tau+2)x^2+\tau^4+6\tau^3+8\tau^2-6$$

$$T(x,\tau)=12\tau(\tau-2)x^2-6\tau^2(\tau-2)(\tau+2)x$$

It is clear that 0 and $\frac{\tau(\tau+2)}{2}$ are solutions of the equation $T(x,\tau)=0$. Now we find the value of S(x; τ) at the point $x=\frac{\tau(\tau+2)}{2}$ and get:

$$W(\tau)=-\frac{\tau^8}{4}-\tau^7+4\tau^5+5\tau^4+6\tau^3+8\tau^2-8$$

It is easy to check that W(0)<0, W(2)>0 and W(3)<0. Let $\tau(\tau_{cr}\approx 2.26)$ be the solution of the equation W(τ) = 0: Then for any value of $\tau\in(2,\tau_{cr})$ there exists only one intersection point of S(x; τ) with the negative x-axis. Consequently, using the fact R(0; τ) > 0 we can conclude there would be no positive roots of the equation R(x; τ) = 0: This completes the proof.

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