SINGULYAR KOEFFISIENTLI ELLIPTIKO – GIPERBOLIK TIPDAGI TENGLAMA UCHUN BIR TRIKOMI-NEYMAN MASALASINING ANALOGI HAQIDA

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Nishonova Sh.T
Muydinjonova B.A

I. Kirish. Yigirmanchi asrning o'rtalaridan boshlab, “aralash tipdagi tenglamalar” deb ataluvchi va gazlar dinamikasi, suyuqliklar dinamikasi, sirtlarning cheksiz kichik bukilish nazariyasi, matematik biologiya va boshqa fan tarmoqlarida ko'plab tatbiqlarga ega bo'lgan differensial tenglamalar jadal o'rganilmoqda. Aralash tipdagi tenglamalar nazariyasida dastlab klassik chegaraviy masalalar, ya'ni soha chegarasining to'liq yoki ba'zi qismlarida funksiyaning qiymati yoki hosilasining (aniq yo'nalishli) qiymati yoki ularning ba'zi kombinatsiyasi berilgan masalalar tadqiq qilingan [1-3]. Mana shunday masalalardan biri Trikomi –Neyman masalasi bo'lib, Ω_0- elliptik sohaning $\overline{\sigma}_0$ egri chiziqda $u(x,y)=\varphi(x,y),(x,y)\in\overline{\sigma}_0$; $\overline{OB}$ chiziqda $u|_{\overline{OB}}=\psi_1(y)$, $0\le y\le 1/\sqrt{2}$; Ω_1- giperbolik sohaning $\overline{OA}$ chiziqda $u(x,y)|_{\overline{OA}}=f(x)$, $0\le x\le\frac{1}{2}$ bilan berilgan shartli masala [1] ishda o'rganilgan. Ushbu maqolada o'rganilayotgan masala Trikomi–Neyman masalasining analogi bo'lib, Ω_0- elliptik sohaning $\overline{\sigma}_0$ egri chiziqda $u(x,y)=\varphi(x,y)$, $(x,y)\in\overline{\sigma}_0$; $\overline{OB}$ chiziqda $\left.\frac{\partial u}{\partial n}\right|_{\overline{OB}}=\psi_2(y)$, $0\le y\le 1/\sqrt{2}$; Ω_1- giperbolik sohaning $\overline{DA}$ chiziqda $u(x,y)|_{\overline{AD}}=f_1(x)$, $\frac{1}{2}\le x\le 1$ shartlar bilan berilgan masaladir [4-5].

II. Masalani qo'yilishi

xOy tekislikda Ω- bir bog'lamli soha, $\sigma_0=\{(x,y):x^2+y^2=1,\ 0<y<x\}$ chegaralangan egri chiziq, va $\overline{OB},\overline{OD},\overline{DA}$ $y=x$, $y=-x$, $y=x-1$ to'g'ri chiziqlar bilan chegaralangan soha berilgan bo'lsin. Bu yerda $O(0,0)$, $A(1,0)$, $B(1/\sqrt{2},1/\sqrt{2}),D(1/2,-1/2)$. Ω sohani $y>0$, $y<0$, $y=0$ mos ravishda Ω_0, Ω_1 OA sohalarga ajratamiz. Ω_0- elliptik soha , Ω_1- giperbolik soha OA- tip o'zgarish chizig'i deyiladi.

$TN^{(2)}$ **masala:** Ω sohada Trikomi masalasini qaraylik.

Ω sohada

$$u_{xx}+signy\cdot u_{yy}+(2\beta/x)u_x+signy(2\beta/y)u_y=0, \qquad (1)$$

(1) tenglamani shunday $u(x,y)\in C(\overline{\Omega})$ regulyar yechimi topilsinki, quyidagi ulash

$$\lim_{y\to -0}(-y)^{2\beta}u_y(x,y)=\lim_{y\to +0}y^{2\beta}u_y(x,y),\ 0<x<1; \quad (2)$$

va chegaraviy

$$u(x,y)=\varphi(x,y),\quad (x,y)\in\overline{\sigma}_0; \quad (3)$$

$$\left.\frac{\partial u}{\partial n}\right|_{\overline{OB}}=\psi_2(y),\quad 0\le y\le 1/\sqrt{2}; \quad (4)$$

$$u(x,y)\big|_{\overline{AD}}=f_1(x),\quad \frac{1}{2}\le x\le 1, \quad (5)$$

shartlarni, hamda $f_1(1)=\varphi(1,0),\ \psi_2(\frac{1}{\sqrt{2}})=\varphi(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ kelishuv shartlarini qanoatlantirsin.

Quyidagi belgilashlarni kiritib olamiz

$$u(x,+0)=u(x,-0)=\tau(x),\quad \tau(x)\in C[0,1]\cap C^2(0,1) \quad (5_1)$$

$$\lim_{y\to -0}(-y)^{2\beta}u_y(x,y)=\nu(x),\quad \nu(x)\in C^2(0,1) \quad (5_2)$$

bu yerda $\varphi(x,y),\psi_2(y),f_1(x)$ -berilgan funksiyalar $\beta=const\in R, 0<\beta<1/2$.

$TN^{(2)}$ **masala yechimining yagonaligi.** Faraz qilaylik masala yechimi ikkita bo'lsin va ular ayirmasini $u(x,y)=u_1(x,y)-u_2(x,y)$ bilan belgilaylik u holda $\varphi(x,y)\equiv\psi_2(y)\equiv\nu(x)\equiv 0$ bo'ladi. Ω_0 sohada $u(x,y)$ funksiya tenglama yechimi ekanligidan foydalanib, $(xy)^{2\beta}u(x,y)$ funksiyani (1) tenglamaga ko'paytirib, divergent holatga keltirib olamiz.

$$\iint_{\Omega_0}(xy)^{2\beta}\left(u_x^2+u_y^2\right)dxdy=\iint_{\Omega_0}\left\{\left[(xy)^{2\beta}uu_x\right]_x+\left[(xy)^{2\beta}uu_y\right]_y\right\}dxdy.$$

Grin-Ostragradiskiy formulasiga asosan (2) shart va $u|_{\overline{\sigma}_0}(x,y)=\lim_{x\to y}\frac{\partial u}{\partial n}=0$, tengliklarga asosan

$$\iint_{\Omega_0}(xy)^{2\beta}\left(u_x^2+u_y^2\right)dxdy=0 \quad (6)$$

bo'ladi. (6) tenglikdan Ω_0 sohada $u(x,y)\equiv const$ tenglik hosil bo'ladi. $\overline{\Omega}_0$ sohada $u(x,y)\in C(\overline{\Omega}_0)$ va $u(x,y)|_{\overline{\sigma}_0}=0$ shartlarga asosan $u(x,y)\equiv 0$ ekanligi kelib chiqadi, bu esa $TN^{(2)}$ masala yechimi yagona ekanligini bildiradi.

Endi masala yechimi mavjudligini ko'rib o'tamiz.

$TN^{(2)}$ **masala yechimi mavjudligi.** Ω_0 sohada $TN^{(2)}$ masala yechimi ko'rinishi quyidagicha yozib olamiz

$$u(x,y)=-\int_0^1 \xi^{2\beta}\nu(\xi)G_3(\xi,0;x,y)d\xi-$$

$$-\sqrt{2}\int_0^{1/\sqrt{2}}\eta^{4\beta}\psi_2(\eta)\frac{\partial}{\partial n}G_3(\eta,\eta;x,y)d\eta+\int_{\sigma 0}(\xi\eta)^{2\beta}\varphi(\xi,\eta)\frac{\partial}{\partial n}G_3(\xi,\eta;x,y)ds.$$

(7)

(7) yechim [2] (1) tenglamani Ω_0 sohada (3) va (4) chegaraviy shartlarni hamda (5.2) belgilashni qanoatlantiruvchi yechim formulasidir.

Bu yerda $s-\sigma_0$ egri chiziq uzunligi, $n-$ ichki normal

$$G_3(\xi,\eta;x,y)=q_1(\xi,\eta;x,y)-\left(r_0^2\right)^{-2\beta}q_1\left(\xi,\eta;x,y\right)+$$

$$+q_2(\xi,\eta;x,y)-\left(r_0^2\right)^{-2\beta}q_2\left(\xi,\eta;x,y\right),$$

$$q_1(\xi,\eta;x,y)=k_1(r_1r_2)^{-2\beta}F(\beta,\beta,2\beta;1-\omega),$$

$$q_2(\xi,\eta;x,y)=k_2(r_1r_2)^{-2\beta}(1-\omega)^{1-2\beta}F(1-\beta,1-\beta,2-2\beta,1-\omega)$$

$$r_0^2=x^2+y^2, x=x/r_0^2, y=y/r_0^2,$$

$$r_j^2=\left[x+(-1)^j\xi\right]^2+\left[y-(-1)^j\eta\right]^2\quad j=\overline{1,2};$$

$$k_1=4^{2\beta-1}\Gamma^2(\beta)/\left[\pi\Gamma(2\beta)\right],$$

$k_2=4^{2\beta-1}\Gamma^2(1-\beta)/\left[\pi\Gamma(2-2\beta)\right]\ 1-\omega=16\xi\eta xy/\left(r_1^2r_2^2\right);$ $F(a,b,c;x)$- Gaussning gipergeometrik funksiyasi []

(7) tenglikdan $y=0$ chiziqdagi qiymatini hisoblab (5₁), (5₂) belgilashlarga asosan $\tau(x)$ va $\nu(x)$ funksiyalar orasidagi funksional munosabatni olamiz.

$$\tau(x)=-\int_0^1\xi^{2\beta}\nu(\xi)G_3(\xi,0;x,0)d\xi-\sqrt{2}\int_0^{1/\sqrt{2}}\eta^{4\beta}\psi(\eta)\frac{\partial}{\partial n}G_3(\eta,\eta;x,0)d\eta+$$

$$+\int_{\sigma 0}(\xi\eta)^{2\beta}\varphi(\xi,\eta)\frac{\partial}{\partial n}G_3(\xi,\eta;x,0)ds,\quad 0\le x\le 1.\qquad (8)$$

(8) tenglikda $x \square x^{1/2}$ almashtirish bajarib va $x^{\beta-1/2}\nu(x^{1/2})=\nu(x)$ belgilashni kiritsak

$$\tau(x^{1/2})=-\frac{k_1}{2}\Gamma(1-2\beta)D_{0x}^{2\beta-1}[\nu(x)]-\frac{k_1}{2}\Gamma(1-2\beta)D_{x1}^{2\beta-1}[\nu(x)]-$$

$$-\sqrt{2}\int_0^{1/\sqrt{2}}\eta^{4\beta}\psi_2(\eta)G_3(\eta,\eta,x^{1/2},0)d\eta+$$

$$+\int_{\sigma_0}(\xi\eta)^{4\beta}\varphi(\xi,\eta)\frac{\partial G_3(\xi,\eta,x^{1/2},0)}{\partial n}ds-$$

$$-\frac{k_1}{2}\int_0^1 \nu(t)[(t+x)^{-2\beta}-(1-tx)^{-2\beta}-(1+tx)^{-2\beta}]dt \qquad (9)$$

tenglik hosil bo'ladi. Bu tenglik Ω_0 sohada $\tau(x)$ va $\nu(x)$ funksiyalar orasidagi funksional munosabatdir. Endi Ω_1 sohada $\tau(x), \nu(x)$ funksiyalar orasida funksional munosabatni olamiz.

$u(x,y)$ - funksiya Ω_1 sohada masalaning yechimi deb faraz qilaylik va (5.1) va (5.2) belgilashlarni e'tiborga olsak, ($\nu(x)$- funksiya $x\to 1$ da $1-2\beta$ kichik tartibda yaqinlashadi) Ω_1 masalaning yechim

$$u(x,y)=\gamma_1\int_0^1 \tau\left(\zeta^{1/2}\right)\left[z(1-z)\right]^{\beta-1}dz-$$

$$-\gamma_2(-xy)^{1-2\beta}\int_0^1 \zeta^{\beta-1/2}\nu\left(\zeta^{1/2}\right)\left[z(1-z)\right]^{-\beta}dz, \qquad (10)$$

ko'rinishda bo'ladi. Bunday masala ko'rinishi o'zgargan Koshi masalasi deyiladi,

bu yerda $\zeta=(x+y)^2-4xyz$, $\gamma_1=\Gamma(2\beta)/\Gamma^2(\beta)$, $\gamma_2=\Gamma(1-2\beta)/\Gamma^2(1-\beta)$;

$\Gamma(z)$- gamma funksiya.

(10) tenglikni (5) shartga bo'ysundiramiz, ya'ni $\overline{DA}$ chiziqdagi ifodasini topamiz.

$$\gamma_1\Gamma(\beta)(1-x)^{1-2\beta}D_{x1}^{-\beta}\left[(1-x)^{\beta-1}\tau\left(x^{1/2}\right)\right]-$$

$$-\gamma_2 4^{2\beta-1}\Gamma(1-\beta)D_{x1}^{\beta-1}\left[x^{\beta-\frac{1}{2}}(1-x)^{-1/2}\nu\left(x^{1/2}\right)\right]=f_1\left[\left(\sqrt{x}+1\right)/2\right], \qquad 0\le x\le 1,$$

(11)

bu yerda D_{xb}^{α} - integrodifferensial operator ko'rinishi [ ]

$$D_{xb}^{\alpha}\varphi(x)=\begin{cases}\dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_x^b (t-x)^{-\alpha-1}\varphi(t)dt \quad agar \quad \alpha<0;\\ \varphi(x), \quad agar \quad \alpha=0;\\ (-1)^n\dfrac{d^n}{dx^n}D_{xb}^{\alpha-n}\varphi(x), \quad agar \quad \alpha>0.\end{cases}$$

(11) tenglikga $(1-x)^{2\beta-1}\Gamma(\beta)/\Gamma(2\beta)$ ifodani ko'paytirib,

$$D_{xb}^{\alpha}D_{xb}^{-\alpha}\varphi(x)=\varphi(x),$$

$$D_{xb}^{\alpha}(1-x)^{2\alpha-1}D_{xb}^{\alpha-1}(1-x)^{-\alpha}f(x)=(1-x)^{\alpha-1}D_{xb}^{2\alpha-1}f(x),$$

tengliklarni e'tiborga olsak,

$$\tau\left(x^{1/2}\right)=\gamma_3 D_{x1}^{2\beta-1}\left[x^{\beta-1/2}\nu\left(x^{1/2}\right)\right]+$$

$$+\gamma_4(1-x)^{1-\beta} D_{x1}^{\beta}\left[(1-x)^{2\beta-1} f_1\left(\frac{\sqrt{x}+1}{2}\right)\right], \ \frac{1}{2}\le x\le 1, \qquad (12)$$

hosil bo'ladi. (12) tenglik Ω_1 sohada $\tau(x)$ va $\nu(x)$ funksiyalar orasidagi funksional munosabatdir.

Endi (9) va (12) tengliklarni e'tiborga olsak,

$$\gamma_5 D_{x1}^{2\beta-1}[\nu(x)]+\frac{k_1}{2}\Gamma(1-2\beta)D_{0x}^{2\beta-1}[\nu(x)]+$$

$$+\frac{k_1}{2}\int_0^1\nu(t)\left[(t+x)^{-2\beta}-(1+tx)^{-2\beta}-|1-tx|^{-2\beta}\right]dt=\Phi(x), \ \ 0<x<1, \qquad (13)$$

bu yerda $\gamma_5=\gamma_3+\gamma_6$, $\gamma_6=k_1\Gamma(1-2\beta)/2$,

$$\Phi(x)=\int_{\sigma_0}(\xi\eta)^{2\beta}\varphi(\xi,\eta)\frac{\partial}{\partial n}G_3\left(\xi,\eta;x^{1/2},0\right)ds-$$

$$-\sqrt{2}\int_0^{1/\sqrt{2}}\eta^{4\beta}\psi(\eta)\frac{\partial}{\partial n}G_3\left(\eta,\eta;x^{1/2},0\right)d\eta$$

$$-\gamma_4(1-x)^{1-\beta}\frac{d}{dx}\int_x^1(1-t)^{2\beta-1}f_1\left(\sqrt{t}+1/2\right)(t-x)^{-2\beta}dt.$$

(13) tenglikga $D_{x1}^{1-2\beta}$ operatorni qo'llab, quyidagi tengliklardan foydalansak,

$$D_{x1}^{1-2\beta}D_{x1}^{2\beta-1}f(x)=f(x),$$

$$D_{x1}^{1-2\beta}D_{0x}^{2\beta-1}f(x)=\cos(1-2\beta)\pi f(x)-\frac{\sin(1-2\beta)\pi}{\pi}\int_0^1\left(\frac{1-t}{1-x}\right)^{1-2\beta}\frac{f(t)}{t-x}dt,$$

$$D_{x1}^{1-2\beta}\left\{\int_0^1 f(t)\left[(1\mp tx)^{-2\beta}\right]dt\right\}=\frac{1}{\Gamma(2\beta)}\int_0^1\left(\frac{1\mp t}{1-x}\right)^{1-2\beta}\frac{f(t)}{1\mp tx}dt, \quad 0<x<1$$

$$D_{x1}^{1-2\beta}\left[\int_0^1 f(t)(t+x)^{-2\beta}dt\right]=\frac{1}{\Gamma(2\beta)}\int_0^1\left(\frac{1+t}{1-x}\right)^{1-2\beta}\frac{f(t)}{t+x}dt, \qquad 0<x<1,$$

hosil bo'ladi.

$$\nu(x)-\frac{\gamma_7}{\pi}\int_0^1\left(\frac{1-t}{1-x}\right)^{1-2\beta}\left[\frac{1}{t-x}+\frac{1}{1-tx}\right]\nu(t)dt+$$

$$+\frac{\gamma_7}{\pi}\int_0^1\left(\frac{1+t}{1-x}\right)^{1-2\beta}\left[\frac{1}{t+x}-\frac{1}{1+tx}\right]\rho(t)dt=\gamma_8\Phi_1(x), \qquad (14)$$

bu yerda $\gamma_7=\cos\beta\pi/(1+\sin\beta\pi)$

$$\Phi_1(x)=2\Gamma(\beta+1/2)\left[(1+\sin\beta\pi)\Gamma(-\beta+1/2)\right]^{-1}D_{x1}^{1-2\beta}\Phi(x),\Phi_1(x)\in C\left[\frac{1}{2},1\right]\cap C^2\left(\frac{1}{2},1\right)$$

(14) ifodada ba'zi shakl almashtirishlarni bajarib, $\xi=2t^2/\left(1+t^4\right), y=2x^2/\left(1+x^4\right)$ belgilashlarni e'tiborga olsak, quyidagi singuliyar integral tenglama hosil bo'ladi.

$$\rho_1(y)-\frac{\gamma_7}{\pi}\int_0^1\frac{\rho_1(\xi)}{\xi-y}d\xi=\Phi_2(y),\qquad 0<y<1, \tag{15}$$

bu yerda

$$\rho_1(y)=(1-x)^{-2\beta}(1+x^4)(1+x)^{-1}\nu(x),$$

$$\Phi_2(y)=\Phi_1(x)-\int_0^1 M(y,\xi)\xi^{-1/2}\rho_1(\xi)d\xi,$$

$$M(y,\xi)=(\gamma_7/2\pi)\frac{(1+x^4)(1+x^2)^{-1}(1+t^4)}{(1+tx)(t+x)(1+\sqrt{1-\xi^2})^{1/2}}\left[\left(\frac{1-t}{1+t}\right)^{2\beta}-1\right].$$

Teorema: Agar $\varphi(x,y)=y^{\alpha}\varphi_0(x,y)$, $\varphi_0(x,y)\in C(\overline{\sigma_0})$, $\alpha>1$; $\psi_2(y)=y^{\delta}\psi_0(y)$, $\psi_0(y)\in C\left[0,1/\sqrt{2}\right]$, $\delta>2$; $f_1(x)=(1-x)^{\gamma}f_0(x)$, $f_0(x)\in C^1\left[\frac{1}{2},1\right]\cap C^2\left(\frac{1}{2},1\right)$, $\gamma>0$ bo'lsa masala yechimi mavjud va yagona bo'ladi.

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