MURAKKAB TIPDAGI UCHINCHI TARTIBLI TENGLAMA UCHUN CHEGARAVIY MASALA

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Annotatsiya: *Mazkur maqolada chegaralanmagan sohada uchinchi tartibli tenglama uchun chegaraviy masala qaralgan bo'lib, masalaning yechimining yagonaligi integral energiya usulida, yechimning mavjudligi esa Fredgolmning ikkinchi tur integral tenglamalari nazariyasidan foydalanilgan holda isbotlangan.*

Kalit so`zlar. *Yuqori tartibli tenglama, Fredgolm integral tenglamasi, yechimning yagonaligi, yechimning mavjudligi, Grin funksiyasi.*

Hozirgi kunda ikkinchi, uchinchi va yuqori tartibli tenglamalar uchun ko`plab mualliflar tomonidan chegaraviy masalalar tahlil etilgan. Murakkab va murakkab-aralash tipdagi tenglamalar uchun ustozlarimiz M.S.Salohiddinov [2], T.D.Jo`rayev [1] va ularning shogirdlari tomonidan chegaraviy masalalar qo`yilib ularni o`rganish nazariyalari yaratilgan. Bu maqolada uchinchi tartibli tenglama uchun chegaralanmagan sohada chegaraviy masala qo`yilgan. Shuni ta'kidlash lozimki, chegaralangan sohada umumiy tenglama uchun chegaraviy masalalar tahlil etilgan [4].

Masalaning qo`yilishi. Ushbu

$$\frac{\partial}{\partial x}\Delta U(x,y)+C(x,y)U(x,y)=0 \qquad (1)$$

tenglamaning

$$D=\{(x,y):0<x<\infty,0<y<\infty\}$$

sohada aniqlangan shunday $U(x,y)$ yechimini topingki, u

1) (1) tenglamaning regulyar yechimi bo`lsin;
2) $\overline{D}$ sohada uzluksiz bo`lsin;
3) quyidagi chegaraviy shartlari qanoatlantirsin:

$$U(x,y)\big|_{x=0}=\phi_1(y), \qquad 0\le y<\infty, \qquad (2)$$

$$U(x,y)\big|_{y=0}=\phi_2(x), \qquad 0\le x<\infty, \qquad (3)$$

$$U_x(x,y)\big|_{x=0}=\phi_3(y), \qquad 0\le y<\infty, \qquad (4)$$

$$\lim_{R\to\infty}U_x(x,y)=0, \quad R^2=x^2+y^2, \qquad x>0,\ y>0. \qquad (5)$$

Bu yerda $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-$ Laplas operatori, $C(x,y)$ va $\phi_i(i=\overline{1,3})$ −berilgan funksiyalar,

$$\phi_1(0)=\phi_2(0) \quad (6)$$

kelishuv sharti o`rinli.

Ushbu ko`rinishda belgilash kiritib,

$$\frac{\partial}{\partial x}U=V \quad (7)$$

(1) tenglamani quyidagi ko'rinishda yozish mumkin:

$$\Delta V+CU=0 \quad (8)$$

(7) ga asosan (2)-(5) chegaraviy shartlardan foydalanib, (8) tenglama uchun quyidagi chegaraviy shartlarni olamiz:

$$V(x,y)\big|_{y=0}=\phi_2'(x), \qquad V(x,y)\big|_{x=0}=\phi_3(y), \quad (9)$$

$$\lim_{R\to\infty}V(x,y)=0, \quad R^2=x^2+y^2, \quad x>0,\ y>0. \quad (10)$$

Masala yechimining yagonaligi. Qo`yilgan masala yechimining yagonaligini ko`rsatish uchun integral energiya usulidan foydalanamiz.

1-teorema . Agar $R\to\infty$ da

1) $\left|C(x,y)\right|\le\frac{C_1}{R^\alpha}, \quad 0<\alpha<1, \quad C_1=const;$

2) $C_x(x,y)\ge0, \quad (x,y)\in D,$

bo`lsa, u holda (1)-(5) masalaning bittadan ortiq yechimi mavjud emas.

Isbot. Bir jinsli masalaning trivial yechimga ega ekanligini ko`rsatamiz, ya'ni

$$\Delta\left(\frac{\partial}{\partial x}U\right)+CU=0 \quad (1)$$

$$U(x,y)\big|_{x=0}=0, \quad U(x,y)\big|_{y=0}=0, \quad U_x(x,y)\big|_{x=0}=0. \quad (11)$$

(1) va (5) masala (7) belgilashga ko`ra quyidagi masalaga ekvivalent:

$$\Delta V+CU=0, \quad (8)$$

$$V(x,y)\big|_{x=0}=0, \quad V(x,y)\big|_{y=0}=0, \quad (12)$$

$$\lim_{R\to\infty}V(x,y)=0, \quad R^2=x^2+y^2, \quad x>0, y>0. \quad (10)$$

Ushbu

$$D_R=\left\{(x,y):x^2+y^2<R^2,x>0,y>0\right\},$$

$$\partial D_R=\left\{(x,y):(x=0)\cup(y=0)\cup\sigma_R\right\},$$

$$\sigma_R=\left\{(x,y):x^2+y^2=R^2,x>0,y>0\right\}$$

sohani qaraymiz.

(8) ko`rinishidagi tenglamani $V(x,y)$ funksiyaga ko`paytirib, D_R soha bo`yicha integral olamiz:

$$\iint_{D_R} V\left(V_{xx}+V_{yy}+CU\right)dxdy=0. \qquad (13)$$

Mazkur ko`paytmada shakl almashtirishlarni bajarib, (10), (12) chegaraviy shartlardan foydalansak, (13) tenglik quyidagi

$$\iint_{D_R}\left[\left(V_x\right)^2+\left(V_y\right)^2+\frac{1}{2}C_xU^2\right]dxdy=0 \qquad (14)$$

ko’rinishga keladi. Bundan,

a) agar $C_x\neq 0$ bo`lsa, u holda (14) dan $\overline{D}$ sohada $U=0$ ekanligi kelib chiqadi.

b) agar $C_x=0$ bo`lsa, u holda (15) dan $V_x=V_y=0$ tenglikni olamiz, $V=const$ bo`ladi. Ammo bir jinsli (14) shartlarga asosan $\overline{D}$ da $V=0$ yoki (7) belgilashga ko`ra $\frac{\partial}{\partial x}U=0$ bo`lishi kelib chiqadi.

So`nggi tenglamaning umumiy yechimi

$$U=\overline{\varphi}(y) \qquad (16)$$

ko’rinishda bo`ladi. Bu yerda $\overline{\varphi}(y)$ –ixtiyoriy noma’lum funksiya. (16) tenglikda bir jinsli (11) chegaraviy shartlardan foydalansak,

$$U(x,y)=0,\ (x,y)\in\overline{D}$$

trivial yechimga ega ekanligini topamiz.

Teorema isbotlandi.

Masala yechimining mavjudligi.

2-teorema . Agar 1-teorema shartlari bajarilsa va

1) $\phi_1(y),\phi_2(x),\phi_3(y)$ funksiyalar $O(0,0)$ nuqta atrofida uzluksiz;

2) $|\phi_1'|\leq\frac{C_1}{y^2},\qquad |\phi_3|\leq\frac{C_3}{y^2},\qquad y\to\infty,$

$|\phi_2|\leq\frac{C_2}{x^2},\qquad x\to\infty$ shartlarni qanoatlantirsa, u holda (1)-(5) masalaning yechimi mavjud.

Isbot. Yechimning mavjudligini isbotlash uchun Grin funksiyalari [3] usulidan foydalanamiz.

$$G(z,\tau)=\frac{1}{2\pi}\ln\left|\frac{z^2-\tau^{-2}}{z^2-\tau^2}\right|,$$

bu yerda $z=x+iy$, $\tau=\xi+i\eta$.

Ma’lumki, Grin funksiyasi

$$\Delta G=0 \qquad (17)$$

tenglamani va

$$G\big|_{\xi=0}=0, \qquad G\big|_{\eta=0}=0 \tag{18}$$

shartlarni qanoatlantiradi.

V funksiya (8)-(10) masalaning yechimi bo`lsin. Biror $(x,y)$ nuqtani $\varepsilon$ radiusli va markazi $(x,y)$ nuqtada bo`lgan C_ε aylana bilan o`raymiz. Aylanadan tashqaridagi sohani $D_R^\varepsilon$ deb belgilaymiz va bu soha uchun Grin formulasini [3] qo`llaymiz.

$$\iint_{D_R^\varepsilon}\left[G(\Delta V-V\Delta G)\right]d\xi d\eta=\int_{\partial D_R}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds \tag{19}$$

(19) tenglikni chap tomoni (17), (18) shartlarga asosan

$$\iint_{D_R^\varepsilon}GCUd\xi d\eta$$

ga teng bo'ladi. O`ng tomonida integral chegarasini almashtirib,

$$\int_{OA\cup\sigma_R\cup OB}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds-\int_{C_\varepsilon}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds=\iint_{D_R^\varepsilon}GCUd\xi d\eta$$

ifodani hosil qilamiz. Ohirgi tenglikda chap tomondagi integralni qaraymiz.

$$I_1=\int_{OA\cup\sigma_R\cup OB}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds=\int_{OA\cup OB}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds+$$

$$+\int_{\sigma_R}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds=-\int_{AO}V(0,\eta)G_\xi(x,y;0,\eta)d\eta+$$

$$+\int_{OB}V(\xi,0)G_\eta(x,y;\xi,0)d\eta+\int_{\sigma_R}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds=$$

$$=\int_0^R G_\xi\phi_3(\eta)d\eta-\int_0^R G_\eta\phi_2'(\xi)d\xi+\int_{\sigma_R}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds,$$

$$I_1=\int_0^R G_\xi\phi_1'(\eta)d\eta-\int_0^R G_\eta\phi_3(\xi)d\xi+\int_{\sigma_R}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds.$$

Bu ifodada $R\to\infty$ da limitga o'tib,

$$I_1=\int_0^R G_\xi\phi_1'(\eta)d\eta-\int_0^R G_\eta\phi_3(\xi)d\xi$$

tenglikni hosil qilamiz.

Keyingi C_ε bo'yicha

$$I_2=\int_{C_\varepsilon}\left(G\frac{\partial V}{\partial n}-V\frac{\partial G}{\partial n}\right)ds$$

integralni hisoblaymiz. Buning uchun Grin funksiyasini quyidagi ko'rinishda yozib olamiz:

$$G(z,\tau)=-\frac{1}{2\pi}\ln|z-\tau|+g(x,y;\xi,\eta)=-\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2}+$$

$$+g(x,y;\xi,\eta)=-\frac{1}{2\pi}\ln r+g(x,y;\xi,\eta).$$

U holda

$$I_{21}=\int_{C_\varepsilon}G\frac{\partial V}{\partial n}ds=\int_{C_\varepsilon}\left[-\frac{1}{2\pi}\ln r+g\right]\frac{\partial V}{\partial n}ds=$$

$$=-\frac{1}{2\pi}\int_{C_\varepsilon}\left[\ln r\cdot\frac{\partial V}{\partial n}(P^*)\cdot 2\pi\varepsilon\right]ds+\int_{C_\varepsilon}g\frac{\partial V}{\partial n}ds.$$

Bu yerda $\frac{\partial V}{\partial n}(P^*)$ – $\frac{\partial V}{\partial n}$ normal hosilaning C_ε aylana ustidagi o’rta qiymati. Bu ifodada $\varepsilon\to 0$ limitga o’tsak,

$$I_{21}=0.$$

Endi I_{22} integralni qaraymiz:

$$I_{22}=\int_{C_\varepsilon}-V\frac{\partial G}{\partial n}ds=-\int_{C_\varepsilon}V\frac{\partial G}{\partial n}ds=-\int_{C_\varepsilon}V\left(-\frac{1}{2\pi}\cdot\frac{1}{r}+\frac{\partial g}{\partial r}\right)ds=$$

$$=\frac{1}{2\pi\varepsilon}\int_{C_\varepsilon}Vds-\int_{C_\varepsilon}V\frac{\partial g}{\partial r}ds=\frac{1}{2\pi\varepsilon}V(P^{**})\cdot 2\pi\varepsilon+\int_{C_\varepsilon}V\frac{\partial g}{\partial r}ds=$$

$$=V(P^{**})+\int_{C_\varepsilon}V\frac{\partial g}{\partial r}ds.$$

Bu yerda P^{**} – chiziq ustidagi ixtiyoriy nuqta. So’nggi ifodada $\varepsilon\to 0$ da limitga o’tib,

$$I_{22}=V(x,y)$$

tenglikni hosil qilamiz. Bularni e’tiborga olsak,

$$I_2=V(x,y).$$

U holda (8)-(10) masalaning yechimini ifodalovchi quyidagi formulaga kelamiz:

$$V(x,y)=\int_0^\infty G_\xi(x,y;0,\eta)\phi_3(\eta)d\eta-\int_0^\infty G_\eta(x,y;\xi,0)\phi_2'(\xi)d\xi-$$

$$-\iint_D G_\xi(x,y;\xi,\eta)C(\xi,\eta)U(\xi,\eta)d\xi d\eta$$

yoki Grin funksiyasini qo’ysak:

$$V(x,y)=\frac{1}{\pi}\int_0^\infty\left(\frac{x}{x^2+(y-\eta)^2}-\frac{x}{x^2+(y+\eta)^2}\right)\phi_3(\eta)d\eta-$$

$$-\frac{1}{\pi}\int_0^\infty\left(\frac{y}{(x-\xi)^2+y^2}-\frac{y}{(x+\xi)^2+y^2}\right)\phi_2'(\xi)d\xi-\iint_D GCUd\xi d\eta.\quad(20)$$

Endi (7) tenglamani (3) shartini qanoatlantiruvchi yechimini qidiramiz. Buning uchun (7) tenglamani $[0,y]$ segmentda integrallab, (20) dan foydalansak,

$$U(x,y)=\frac{1}{\pi}\int_0^x\left[\int_0^\infty\left(\frac{x}{x^2+(t-\eta)^2}-\frac{x}{x^2+(t+\eta)^2}\right)\phi_3(\eta)d\eta\right]dt-$$

$$-\frac{1}{\pi}\int_0^\infty\left(\frac{t}{(x-\xi)^2+t^2}-\frac{t}{(x+\xi)^2+t^2}\right)\phi_2'(\xi)d\xi-\iint_D GCUd\xi d\eta+\phi_1(y). \qquad (21)$$

(21) tenglikda integrallarni hisoblash natijasida quyidagi

$$U(x,y)+\iint_D K(x,y;\xi,\eta)U(\xi,\eta)d\xi d\eta=F(x,y) \qquad (22)$$

ko`rinishdagi tenglamaga kelamiz. Bu yerda

$$K(x,y;\xi,\eta)=C(\xi,\eta)\int_0^y G(x,t;\xi,\eta)dt\,,$$

$$F(x,y)=-\int_0^\infty G_1(x,y;0,\eta)\phi_3(\eta)d\eta+\int_0^\infty G_2(x,y;\xi,0)\phi_2'(\xi)d\xi+\phi_1(y),$$

$$G_1(x,y;0,\eta)=arctg\frac{x-\eta}{x}-arctg\frac{x+\eta}{x},$$

$$G(x,y;\xi,0)=\frac{1}{2}\left[\ln\left(x^2+(x-\xi)^2\right)-\ln\left(x^2+(x+\xi)^2\right)+2\ln\frac{x+\xi}{x-\xi}\right].$$

(22) tenglama Fredgolmning ikkinchi tur integral tenglamasi bo`lib, yechimning mavjudligi yagonalik teoremasidan kelib chiqadi.

Mavjudlik teoremasi shartlari bajarilganda $U(x,y)$ funksiya masalada qo`yilgan barcha shartlarni bajaradi.

Teorema isbotlandi.

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