THE TWO-PARAMETRIC MITTAG-LEFFLER FUNCTION AND ITS SOME PROPERTIES

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Abstract: *In this article it is presented that the basic properties of the two-parametric Mittag-Leffler function* $E_{\alpha,\beta}(z)$*, which is the most straightforward generalization of the classical Mittag-Leffler function* $E_{\alpha}(z)$*. Its order and type are calculated, this article presents a number of formulas relating to the two-parametric Mittag-Leffler function's recurrence relations and differentiation, introduce some useful integral representations.*

Keywords: *Gamma function, complementary error function, Prabhakar function, Abel integral equation, Beta function.*

1. Series representation

The Mittag-Leffler type function (or the *two-parametric Mittag-Leffler function*)

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\ (\operatorname{Re}\alpha>0,\ \beta\in\mathbb{C})\tag{1}$$

Generalizes the classical Mittag-Leffler function

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}\ (\operatorname{Re}\alpha>0).$$

We have $E_{\alpha,1}(z)=E_{\alpha}(z)$. For any $\alpha,\beta\in\mathbb{C}$, $\operatorname{Re}\alpha>0$, the function (1) is an entire function of order $\rho=1/(\operatorname{Re}\alpha)$ and type $\sigma=1$.

Indeed, let us consider the slightly more general function

$$E_{\alpha}(\sigma^{\alpha}z)=\sum_{k=0}^{\infty}\frac{(\sigma^{\alpha}z)^k}{\Gamma(\alpha k+1)},\tag{2}$$

i.e. take the coefficients in the form

$$c_k=\frac{\sigma^{\alpha k}}{\Gamma(\alpha k+\beta)}\ (k=0,1,2,\dots)\tag{3}$$

Where $0<\operatorname{Re}\alpha<+\infty,0<\sigma<+\infty$ is an arbitrary real constant and β is a complex parameter. By using Stirling's formula we have

$$\Gamma(\alpha k+\beta)=\sqrt{2\pi}(\alpha k)^{\alpha k+\beta-\frac{1}{2}}e^{-\alpha k}\left[1+o(1)\right],\ k\to\infty\tag{4}$$

And, consequently, for the sequence $\{c_k\}_0^{\infty}$ we immediately obtain

$$\lim_{k\to\infty}\frac{k\log k}{\log\frac{1}{|c_k|}}=\frac{1}{(\operatorname{Re}\alpha)}=\rho,\quad \lim_{k\to\infty}k|c_k|^{\rho/k}=e\rho\sigma\tag{5}$$

2. Differential and Recurrence Relations

A term-by-term differentiation allows us to verify in an easy way that

$$\frac{d^m}{dz^m}E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{\Gamma(k+m+1)z^k}{k!\Gamma(\alpha k+\alpha m+\beta)}=m!E_{\alpha,\alpha m+\beta}^{m+1}(z)\quad(m\ge 1)\qquad(6)$$

Where $E_{\alpha,\beta}^{\gamma}(z)$ denotes the 3-parametric Mittag-Leffler function (also known as the Prabhakar function).

The following formula expresses the first derivative of the ML function in terms of the difference of two instances of the same function

$$\frac{d}{dz}E_{\alpha,\beta}(z)=\frac{E_{\alpha,\beta-1}(z)+(1-\beta)E_{\alpha,\beta}(z)}{\alpha z},\ z\ne 0\qquad(7)$$

It can be generalized to derivatives of any integer order, and thus provides a summation formula of Djrbashian type: *Let* $\alpha>0,\beta\in\square$ *and* $z\ne 0$. For any $m\in\square$

$$\frac{d^m}{dz^m}E_{\alpha,\beta}(z)=\frac{1}{\alpha^m z^m}\sum_{j=0}^{m}c_j^{(m)}E_{\alpha,\beta-j}(z)\qquad(8)$$

where $c_0^{(0)}=1$ *and the remaining coefficients* $c_j^{(m)},\ j=0,1,\ldots,m$*, are recursively evaluated as*

$$c_j^{(m)}=\begin{cases}(1-\beta-\alpha(m-1))c_0^{(m-1)} & j=0,\\ c_{j-1}^{(m-1)}+(1-\beta-\alpha(m-1)+j)c_j^{(m-1)} & 1\le j\le m-1,\\ 1 & j=m\end{cases}\qquad(9)$$

Several applications involve the more general function $z^{\beta-1}E_{\alpha,\beta}(z^{\alpha})$, for which differentiation formulas can easily be derived. The following differentiation formula is an immediate consequence of the definition of the two-parametric Mittag-Leffler function (1)

$$\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}E_{\alpha,\beta}(z^{\alpha})\right]=z^{\beta-m-1}E_{\alpha,\beta-m}(z^{\alpha})\quad(m\ge 1)\qquad(10)$$

We consider now some corollaries of formula (10). Let $\alpha=m/n,\ (m,n=1,2,\ldots)$ in (10). Then

$$\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}E_{m/n,\beta}(z^{m/n})\right]=$$

$$=z^{\beta-1}E_{m/n,\beta}(z^{m/n})+z^{\beta-1}\sum_{k=1}^{n}\frac{z^{-\frac{m}{n}k}}{\Gamma\left(\beta-\frac{m}{n}k\right)}\quad(m,n\ge 1)\qquad(11)$$

Since

$$\frac{1}{\Gamma(-s)}=0\quad(s=0,1,2,\ldots),$$

It follows from (11) with n=1 and $\beta=0,1,\ldots,m$ that

$$\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}E_{m,\beta}\left(z^m\right)\right]=z^{\beta-1}E_{m,\beta}\left(z^m\right)\quad(m\ge 1)\tag{12}$$

Substituting $z^{n/m}$ in place of z in (11) we get

$$\left(\frac{m}{n}z^{1-\frac{n}{m}}\frac{d}{dz}\right)^m\left[z^{(\beta-1)\frac{n}{m}}E_{m,\beta}(z)\right]$$
$$=z^{(\beta-1)\frac{n}{m}}E_{m,\beta}(z)+z^{(\beta-1)\frac{n}{m}}\sum_{k=1}^{n}\frac{z^{-k}}{\Gamma\left(\beta-\frac{m}{n}k\right)}\qquad(m,n=1,2,\ldots)\tag{13}$$

Let $m=1$ in this formula. We then obtain the first-order differential equation for the function $z^{(\beta-1)n}E_{1/n,\beta}(z)$:

$$\frac{1}{n}\frac{d}{dz}\left[z^{(\beta-1)n}E_{1/n,\beta}(z)\right]-z^{n-1}\left[z^{(\beta-1)n}E_{1/n,\beta}(z)\right]$$
$$=z^{\beta n-1}\sum_{k=1}^{n}\frac{z^{-k}}{\Gamma\left(\beta-\frac{k}{n}\right)},\quad(n=1,2,\ldots)\tag{14}$$

Here we consider $u=z^{(\beta-1)n}E_{1/n,\beta}(z)$; $f(z)=z^{\beta n-1}\sum_{k=1}^{n}\frac{z^{-k}}{\Gamma\left(\beta-\frac{k}{n}\right)}$ in order to solve the non-homogeneous ordinary differential equation, and then

$$\frac{1}{n}\frac{du}{dz}-z^{n-1}u=f(z)$$

Solving this equation, we obtain for any $z_0\ne 0$,

$$E_{1/n,\beta}(z)=z^{(1-\beta)n}e^{z^n}\left\{z_0^{(\beta-1)n}e^{-z_0^n}E_{1/n,\beta}(z_0)+n\int_{z_0}^{z}e^{-\tau^n}\left(\sum_{k=1}^{n}\frac{\tau^{-k}}{\Gamma\left(\beta-\frac{k}{n}\right)}\tau^{\beta n-1}\right)d\tau\right\}\quad(n=1,2,\ldots)\tag{15}$$

Formula (15) is true with $z_0=0$ if $\beta=1$. In this case we have

$$E_{1/n,1}(z)=e^{z^n}\left\{1+n\int_0^z e^{-\tau^n}\left(\sum_{k=1}^{n-1}\frac{\tau^{k-1}}{\Gamma\left(\frac{k}{n}\right)}\right)d\tau\right\}(n\ge 2)\tag{16}$$

In particular,

$$E_{1/2,1}(z)=e^{z^2}\left\{1+\frac{2}{\sqrt{\pi}}\int_0^z e^{-\tau^2}d\tau\right\}=e^{z^2}\{1+\operatorname{erf} z\}=e^{z^2}\operatorname{erfc}(-z)\tag{17}$$

And, consequently,

$$E_{1/2,1}(z) \sim 2e^{z^2},\ |\arg z| < \frac{\pi}{4},\ |z| \to \infty$$

In the following lemma we collect together a number of known recurrence relations for the two-parametric Mittag-Leffler function.

Lemma 1 *For all* $\alpha > 0, \beta > 0$ *the following relations hold*

$$z^2 E_{\alpha,\beta+2\alpha}(z) = E_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)} - \frac{z}{\Gamma(\beta+\alpha)}$$

(18)

$$z^3 E_{\alpha,\beta+3\alpha}(z) = E_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)} - \frac{z}{\Gamma(\beta+\alpha)} - \frac{z^2}{\Gamma(\beta+2\alpha)}$$

(19)

$$z^4 E_{\alpha,\beta+4\alpha}(z) = E_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)} - \frac{z}{\Gamma(\beta+\alpha)} - \frac{z^2}{\Gamma(\beta+2\alpha)} - \frac{z^3}{\Gamma(\beta+3\alpha)}$$

(20)

$\triangleleft$ Let us prove one of these relations:

$$z^2 E_{\alpha,\beta+2\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^{k+2}}{\Gamma(\alpha k+\beta+2\alpha)} = \begin{Bmatrix} k+2=n \\ k=0 \to n=2 \\ k \to \infty \to n \to \infty \end{Bmatrix}$$

$$= \sum_{n=2}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}$$

For returning the form of two-parametric Mittag-Leffler function, we need initial two elements in the last series, so we will change it a bit

$$z^2 E_{\alpha,\beta+2\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} - \frac{1}{\Gamma(\beta)} - \frac{z}{\Gamma(\alpha+\beta)}.$$

3. Integral Relations

Using the well-known discrete orthogonality relation

$$\sum_{h=0}^{m-1} e^{i2\pi hk/m} = \begin{cases} m, & \text{if } k \equiv 0 \pmod m \\ 0, & \text{if } k \not\equiv 0 \pmod m \end{cases}$$

And definition (1) of the function $E_{\alpha.\beta}(z)$ we have

$$\sum_{h=0}^{m-1} E_{\alpha,\beta}\left(ze^{i2\pi hk/m}\right) = mE_{\alpha/m,\beta}\left(z^m\right) \quad (m \ge 1) \qquad (21)$$

Substituting here $m\alpha$ for α and $z^{1/m}$ for z we obtain

$$E_{\alpha,\beta}(z) = \frac{1}{m}\sum_{h=0}^{m-1} E_{m\alpha,\beta}\left(z^{1/m}e^{i2\pi h/m}\right) \quad (m \ge 1). \qquad (22)$$

Similarly, the formula

$$E_{\alpha,\beta}(z)=\frac{1}{2m+1}\sum_{h=-m}^{m}E_{(2m+1)\alpha,\beta}\left(z^{1/(2m+1)}e^{i2\pi h/(2m+1)}\right)\ (m\geq 0) \tag{23}$$

can be obtained via the relation

$$\sum_{h=-m}^{m} e e^{i2\pi hk/(2m+1)}=\begin{cases}2m+1, & \text{if}\quad k\equiv 0 \quad (\mathrm{mod}\ 2m+1)\,,\\ 0, & \text{if}\quad k\not\equiv 0\quad (\mathrm{mod}\ 2m+1)\,.\end{cases}$$

Using (1) and term-by-term integration we arrive at

$$\int_0^z E_{\alpha,\beta}\left(\lambda t^{\alpha}\right)t^{\beta-1}dt=z^{\beta}E_{\alpha,\beta+1}\left(\lambda z^{\alpha}\right)\ (\beta>0) \tag{24}$$

And furthermore, at the more general relation

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\mu-1}E_{\alpha,\beta}\left(\lambda t^{\alpha}\right)t^{\beta-1}dt = z^{\mu+\beta-1}E_{\alpha,\mu+\beta}\left(\lambda z^{\alpha}\right)\ (\mu>0,\beta>0) \tag{25}$$

Where the integration is performed along the straight line connecting the points 0 and z .

It follows from formulas (25), (22) and (23) that

$$\frac{1}{\Gamma(\beta)}\int_0^z (z-t)^{\beta-1}e^{\lambda t}dt=z^{\beta}E_{1,\beta+1}(\lambda z)\ (\beta>0) \tag{26}$$

$$\frac{1}{\Gamma(\beta)}\int_0^z (z-t)^{\beta-1}\cosh\sqrt{\lambda}t dt=z^{\beta}E_{2,\beta+1}\left(\lambda z^2\right)(\beta>0) \tag{27}$$

$$\frac{1}{\Gamma(\beta)}\int_0^z (z-t)^{\beta-1}\frac{\sinh\sqrt{\lambda}t}{\sqrt{\lambda}}dt=z^{\beta+1}E_{2,\beta+2}\left(\lambda z^2\right)\ (\beta>0). \tag{28}$$

Here we use series representations of the functions, $e^{\lambda t}$, $\cosh\sqrt{\lambda}t$, $\sinh\sqrt{\lambda}t$ in order to evaluate the integrals above.

◁ Let us prove the relation

$$z^{\beta-1}E_{\alpha,\beta}\left(z^{\alpha}\right)=z^{\beta-1}E_{2\alpha,\beta}\left(z^{2\alpha}\right)+ +\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}E_{2\alpha,\beta}\left(t^{2\alpha}\right)t^{\beta-1}dt\ \ (\beta>0) \tag{29}$$

First of all, we have by direct evaluations

$$\int_0^z E_{2\alpha,\beta}\left(t^{2\alpha}\right)t^{\beta-1}\left\{1+\frac{(z-t)^{\alpha}}{\Gamma(\alpha+1)}\right\}dt=$$

$$=\sum_{k=0}^{\infty}\frac{1}{\Gamma(2k\alpha+\beta)}\int_0^z t^{2k\alpha+\beta-1}\left\{1+\frac{(z-t)^{\alpha}}{\Gamma(\alpha+1)}\right\}dt$$

$$=z^{\beta}\sum_{k=0}^{\infty}\frac{z^{2k\alpha}}{\Gamma(2k\alpha+\beta+1)}+z^{\beta}\sum_{k=0}^{\infty}\frac{z^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+\beta+1)}$$

$$=z^{\beta}\sum_{k=0}^{\infty}\frac{z^{k\alpha}}{\Gamma(k\alpha+\beta+1)}=z^{\beta}E_{\alpha,\beta+1}\left(z^{\alpha}\right)$$

This relations and formula (24) imply

$$\int_0^z E_{2\alpha,\beta}\left(t^{2\alpha}\right)t^{\beta-1}\left\{1+\frac{(z-t)^{\alpha}}{\Gamma(\alpha+1)}\right\}dt$$

$$=\int_0^z E_{\alpha,\beta}\left(t^{\alpha}\right)t^{\beta+1}dt\quad(\beta>0)$$

Differentiating of this formula with respect to z gives us formula (29).

◁ Let us prove the formula

$$\int_0^l z^{\beta-1}E_{\alpha,\beta}\left(\lambda x^{\alpha}\right)(l-x)^{\nu-1}E_{\alpha,\nu}\left(\lambda^*(l-x)^{\alpha}\right)dx$$

$$=\frac{\lambda E_{\alpha,\beta+\nu}\left(l^{\alpha}\lambda\right)-\lambda^* E_{\alpha,\beta+\nu}\left(l^{\alpha}\lambda^*\right)}{\lambda-\lambda^*}l^{\beta+\nu-1}\quad(\beta>,\nu>0) \qquad (30)$$

Where λ and $\lambda^*\left(\lambda\neq\lambda^*\right)$ are any complex parameters.

Indeed, using (1) for any λ, $\lambda^*\left(\lambda\neq\lambda^*\right)$ and $\beta>0,\ \nu>0$ we find

$$\int_0^l x^{\beta-1}E_{\alpha,\beta}(\lambda x^{\alpha})(l-x)^{\nu-1}E_{\alpha,\nu}(\lambda^*(l-x)^{\alpha})\,dx$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\lambda^n(\lambda^*)^m}{\Gamma(n\alpha+\beta)\,\Gamma(m\alpha+\nu)}\int_0^l x^{n\alpha+\beta-1}(l-x)^{m\alpha+\nu-1}\,dx$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\lambda^n(\lambda^*)^m l^{(n+m)\alpha+\beta+\nu-1}}{\Gamma((m+n)\alpha)+\beta+\nu)}=l^{\beta+\nu-1}\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{\lambda^n(\lambda^*)^{k-n}\,l^{k\alpha}}{\Gamma(k\alpha+\beta+\nu)}$$

$$=l^{\beta+\nu-1}\sum_{k=0}^{\infty}\frac{(\lambda^*)^k\,l^{k\alpha}}{\Gamma(k\alpha+\beta+\nu)}\sum_{n=0}^{k}\left(\frac{\lambda}{\lambda^*}\right)^n=\frac{l^{\beta+\nu-1}}{\lambda-\lambda^*}\sum_{k=0}^{\infty}\frac{l^{k\alpha}(\lambda^{k+1}-(\lambda^*)^{k+1})}{\Gamma(k\alpha+\beta+\nu)}.$$

Using formula (1) once more, we arrive at (30).

Finally, we obtain two integral relations:

$$\int_0^{+\infty} e^{-t} E_{\alpha,\beta}(zt^{\alpha}) t^{\beta-1} dt = \frac{1}{1-z} \quad (\beta > 0,\ |z| < 1), \tag{31}$$

$$\int_0^{+\infty} e^{-t^2/(4x)} E_{\alpha,\beta}(t^{\alpha}) t^{\beta-1} dt = \sqrt{\pi} x^{\beta/2} E_{2\alpha,\frac{1+\beta}{2}}(x^{2\alpha}) \quad (\beta > 0,\ x > 0).$$

(32)

First of all, since the Mittag-Leffler function (1) is an entire function of order $\rho = 1/(\mathrm{Re}\,\alpha)$ and type $\sigma = 1$, we have for any $\sigma > 1$ the estimate:

$$\left|E_{\alpha,\beta}(z)\right| \le Ce^{\sigma|z|^{\rho}},\ |z| \ge |z_{\sigma}|.$$

Consequently, the integrals in the formulas (31) and (32) are convergent.

Mittag-Leffler type function $E_{\alpha,\alpha}(z)$ plays an essential role in the linear Abel integral equation of the second kind.

Theorem 1. *Let a function $f(t)$ be in the function space $L_1(0,l)$. Let $\alpha > 0$ and λ be an arbitrary complex parameter. Then the integral equation*

$$u(t) = f(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau,\ t \in (0,l) \tag{33}$$

has a unique solution

$$u(t) = f(t) + \lambda \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda(t-\tau)^{\alpha}\right] f(\tau) d\tau,\ t \in (0,l) \tag{34}$$

in the space $L_1(0,l)$.

This result was discovered in the pioneering work of Hille and Tamarkin.

◁ *Proof.* According to a course of Integral Equations, this is a particular form of Volterra integral equation. In our case, the kernel of the given integral equation is equal to $K(t,\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$. In order to find the solution of this integral equation, we can use iterative kernels, which are denoted below:

$$K_1(t,\tau) = K(t,\tau)$$

$$K_{n+1}(t,\tau) = \int_{\tau}^{t} K(t,z) K_n(z,\tau) dz \quad (n = 1,2,...)$$

Then we can easily find the solution with the means of resolvent of this integral equation,

$$u(t) = f(t) + \lambda \int_0^t R(t,\tau;\lambda) f(\tau) d\tau$$

The resolvent is denoted

$$R(t,\tau;\lambda) = \sum_{\nu=0}^{\infty} \lambda^{\nu} K_{\nu+1}(t,\tau)$$

In this case, it is not easy to notice that the resolvent is equal to

$$R(t,\tau;\lambda)=(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left[\lambda(t-\tau)^{\alpha}\right]$$

4. CONCLUSION

The two-parameteric Mittag-Leffler function $E_{\alpha,\beta}$ has a fundamental importance in fractional calculus, and it appears frequently in the solutions of fractional differential and integral equations. However, the expense of calculating this function often prompts efforts to devise accurate approximations that are more cost-effective.

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