# ON THE CENTRAL LIMIT THEOREM FOR DEPENDENT RANDOM VARIABLES

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**Abstract:** *In this article, we proved central limit theorem for dependent random variables.*
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**Theorem**. Let $\{X_{n,i}\}$ be a triangular array of mean zero random variables. For each $n=1,2,\ldots$ let $d=d_n$, $m=m_n$ and suppose $X_{n,1},\ldots,X_{n,d}$ is an m-dependent sequence of random variables. Define

$$B_{n,k,a}^2 = Var\left(\sum_{i=a}^{a+k-1} X_{n,i}\right),$$

$$B_n^2 \equiv B_{n,d,1} \equiv Var\left(\sum_{i=1}^{d} X_{n,i}\right).$$

Assume the following conditions hold. For some $\delta > 0$ and some $-1 \le \gamma < 1$:

$$E\left|X_{n,i}\right|^{2+\delta} \le \Delta_n \text{ for all } i, \quad (1)$$

$$B_{n,k,a}^2 / (k^{1+\gamma}) \le K_n \text{ for all } a \text{ and for all } k \ge m, \quad (2)$$

$$B_n^2 / (dm^{\gamma+1}) \ge L_n, \quad (3)$$

$$K_n / L_n = O(1), \quad (4)$$

$$\Delta_n / L_n^{(2+\delta)/2} = O(1) \quad (5)$$

$$m^{2+(1-\gamma)(1+2/\delta)} / d \to 0 \quad (6)$$

Then, $B_n^{-1}(X_{n,1}+\ldots+X_{n,d}) \Rightarrow N(0,1)$

**Proof of Theorem** . In the proof we will need a result for bounding moments of m-dependent sequences. We will state it as a corollary of the following lemma, which implicitly is given in Chow and Teicher (1978) and deals with independent sequences.

**Lemma A.1.** Let $\{Y_i\}$ be an independent sequence of mean zero random variables. Assume $E\left|Y_i\right|^q \le \Delta$ for some $q \ge 2$ and all $i$.

Then, $$E\left|\sum_{i=1}^{n} Y_i\right|^q \le C_q^q \Delta n^{q/2}$$

Where $C_q$ is a positive constant depending only upon $q$.

Proof. See Theorem 2 and Corollary 2 in Section 10:3 of Chow and Teicher (1978).

**Corollary A.1.** Let $\{X_i\}$ be an m-dependent sequence of mean zero random variables. Assume $E\left|X_i\right|^q \le \Delta$ for some $q \ge 2$ and all $i$.

Then, for all $n \geq 2m$,

$$E\left|\sum_{i=1}^{n} X_i\right|^q \leq C_q^q \Delta(4mn)^{q/2}$$

where $C_q$ is a positive constant depending only upon $q$.

**Proof.** Define $t = [n/m]$ where $[\cdot]$ denotes the integer part. Now split $X_1 + \ldots + X_n$ into $t$ blocks of size $m$ and a remainder block: $X_1 + \ldots + X_n \equiv A_1 + \ldots + A_t + A_{t+1}$ Due to $m$-dependence, the odd-numbered blocks are independent of each other, as are the even-numbered blocks. This allows us to apply Lemma A.1:

$$\left\|\sum_{i=1}^{n} X_i\right\|_q \leq \left\|\sum_{i \text{ odd}} A_i\right\|_q + \left\|\sum_{i \text{ even}} A_i\right\|_q \quad \text{(by Minkowski)}$$

$$\leq 2C_q m(\Delta)^{1/q}(t/2+1)^{1/2} \quad \text{(by Lemma A:1 and Minkowski):}$$

But, this is equivalent to

$$E\left|\sum_{i=1}^{n} X_i\right|^q \leq C_q^q 2^q m^q \Delta(t/2+1)^{q/2}$$

$$\leq C_q^q 2^q m^q \Delta(t)^{q/2} \leq C_q^q 2^q \Delta(mn)^{q/2} = C_q^q \Delta(4mn)^{q/2}.$$

We are now able to prove the theorem. The main idea of the proof follows Berk (1973), but we need some modifications, since our theorem is more general.

For each $n$, we choose an integer $p = p_n > 2m$ so that

$$\lim_{n\to\infty} m/p = 0, \qquad \lim_{n\to\infty} p^{1+(1-\gamma)(1+2/\delta)}/d = 0. \qquad (7)$$

This can be done, for example, by remembering assumption (6) and choosing p to be the smallest integer greater than $2m$ and greater than $m^{1/2}d^{1/2\xi}$, where $\xi$ is equal to $1+(1-\gamma)(1+2/\delta)$. Next, define integers $t = t_n$ and $q = q_n$ by $d = pt + q$, $0 \leq q < p$. The main idea of the proof is to split the sum $X_{n,1} + \ldots + X_{n,d}$ into alternate blocks of length $p - m$ (the big blocks) and $m$ (the little blocks). This is a common approach to proving central limit theorems for dependent random variables, and is attributed to Markov in Bernstein (1927). Let

$$U_{n,i} = X_{n,(i-1)p+1} + \ldots + X_{n,ip-m}, \; 1 \leq i \leq t,$$

$$V_{n,i} = X_{n,ip-m+1} + \ldots + X_{n,ip}, \quad 1 \leq i \leq t$$

$$U_{n,t+i} = X_{n,tp+1} + \ldots + X_{n,d}.$$

By definition, $X_{n,1} + \ldots + X_{n,d} = \sum_{i=1}^{t+1} U_{n,i} + \sum_{i=1}^{t} V_{n,i}$. Since the $X_{n,i}$ are $m$-dependent and $p > 2m$, $\{U_{n,i}\}$ and $\{V_{n,i}\}$ are each independent sequences. It is easily seen that the difference between $B_n^{-1}(X_{n,1} + \ldots + X_{n,d})$ and has variance approaching zero. Indeed,

$$Var\left(B_n^{-1}\sum_{i=1}^{t+1}V_{n,i}\right)=B_n^{-2}\sum_{i=1}^{l}Var(V_{n,i})$$

$$\le B_n^{-2}t\left[\sup_i \; Var(V_{n,i})\right]\le B_n^{-2}tK_n m^{1+\gamma} \quad \text{(by assumption (2))}$$

$$\le B_n^{-2}(d/p)K_n m^{1+\gamma}$$

$$\le \frac{K_n}{L_n}\frac{m}{n}\to 0 \quad \text{(by assumption (3) and (4)).}$$

Hence, provided they exist, the asymptotic distributions of the two quantities $B_n^{-1}\sum_{i=1}^{t+1}U_{n,i}$ and $B_n^{-1}\sum_{i=1}^{d}X_{n,i}$ are the same, and the goal now is to show that

$$B_n^{-1}\sum_{i=1}^{t+1}U_{n,i}\Rightarrow N(0,1).$$

In order to apply assumption (3) again, we will first establish that $B_n^{-2}Var\left(\sum_{i=1}^{t+1}U_{n,i}\right)$ tends to one, or, equivalently, $B_n^{-2}Cov\left(\sum_{i=1}^{t+1}U_{n,i},\sum_{i=1}^{t}V_{n,i}\right)$ tends to zero. Note first that $Cov\left(U_{n,i},V_{n,i}\right)=0$ unless $j=i$ or $i-1$. Furthermore,

$$\left|Cov\left(U_{n,i},V_{n,i}\right)\right|=\left|E\left(U_{n,i},V_{n,i}\right)\right|\le\left[Var(U_{n,i})Var(V_{n,i})\right]^{1/2}$$

$$\le K_n(mp)^{(1+\gamma)/2} \quad \text{(by assumption (2)).}$$

Combining these two facts, we obtain $\left|Cov\left(\sum_{i=1}^{t+1}U_{n,i},\sum_{i=1}^{t}V_{n,i}\right)\right|\le 2K_n(mp)^{(1+\gamma)/2}$ and finally,

$$B_n^{-2}Cov\left(\sum_{i=1}^{t+1}U_{n,i},\sum_{i=1}^{t}V_{n,i}\right)\le 2\frac{K_n}{L_n}\frac{t}{dm^{\gamma}}(mp)^{(1+\gamma)/2}$$

$$\le 2\frac{K_n}{L_n}\frac{1}{pm^{\gamma}}(mp)^{(1+\gamma)/2}$$

$$=2\frac{K_n}{L_n}\left(\frac{m}{p}\right)^{(1-\gamma)/2}\to 0 \quad \text{(by assumption (4) and since } \gamma<1).$$

By Lyapounov's theorem, it will now suffice to verify that $\sum_{i=1}^{t+1}E\left|U_{n,i}\right|^{2+\delta}/B_n^{2+\delta}$ tends to zero. By Corollary A.1,

$$E\left|U_{n,i}\right|^{2+\delta}\le C_{2+\delta}^{2+\delta}\Delta_n(4pm)^{(2+\delta)/2},\quad 1\le i\le t+1,$$

And therefore

$$\sum_{i=1}^{t+1}E\left|U_{n,i}\right|^{2+\delta}/B_n^{2+\delta}\le Const.\Delta_n(d/p+1)(pm)^{(2+\delta)/2}/B_n^{2+\delta}$$

By assumption (3), finally,

$$\Delta_n(d/p)(pm)^{(2+\delta)/2}/B_n^{2+\delta}\le\Delta_n L_n^{-(2+\delta)/2}\frac{d}{p}\left(\frac{pm}{dm^{\gamma}}\right)^{(2+\delta)/2}$$

$$\leq \Delta_n L_n^{-(2+\delta)/2} \left(\frac{p}{d}\right)^{\delta/2} m^{(1-\gamma)(2+\delta)/2}$$

$$= O(1) AB \quad \text{(by assumption (5));}$$

where $A = p^{\delta/2+(1-\gamma)(2+\delta)/2} d^{-\delta/2}$ and $B = \left(\frac{m}{p}\right)^{(1-\gamma)(2+\delta)/2}$. The second condition on $p$ in (7) implies that $A$ tends to zero. The first condition on $p$ in (7), together with the fact that $\gamma \leq 1$, imply that $B$ tends to zero as well.

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